Universidad Autónoma de Baja California
Facultad de Ingeniería, Arquitectura y Diseño

INGENIERÍA EN NANOTECNOLOGÍA

ETAPA DISCIPLINARIA

APUNTES

13185 TEORÍA ELECTROMAGNÉTICA

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ELECTROSTATIC FIELDS

Introduction

Before we commence our study of electrostatics, it might be helpful to examine briefly the importance of such study. Electrostatics is fascinating subject that has grown up in diverse areas of application.

- Electric power transmission, X-ray machine, and lightning protection are associate with strong electric field and will required a knowledge of electrostatics to understand and design suitable equipment.

- The device used in solid-sate electronic are based on electrostatic. These include resistors, capacitors, and device such as bipolar and field effect transistors, which are based on control of electron motion by electrostatic fields.

- Almost all peripheral devices, with de exception of magnetic memory, are based on electrostatic fields. Toch pad, capacitance keyboards, chatode-ray tube, liquid crystal display, and electrostatics printers are typical examples.

- In medical work, diagnosis is often carried out with the aid of electrostatic, as incorporated in electrocardiograms, electroencephalograms, and other recording of the electrical activity of organs include eyes, ears, and stomach.

- In industry, electrostatics is applied in a variety of forms such as paint spraying, electrodeposition, electrochemical machining, and separation of fine particles.

- Electrostatics is used in agriculture to sort seed, for direct spraying of plant, to mesure te moisture content in crops, to spin cotton, and for speed baking bread and smoking meat.

Coulomb's law and Field Intensity

- Coulomb's law is and experimental law formulated in 1785 by Charles Augustin de Coulomb, then colonel in the France army.

- One coulomb is approximately equivalent to 6×10^{18} electron; it is very large unit of charge because one electron charge $e = -1.6019 \times 10^{-19}$ C.

- Coulomb's law states that the Force F between two carge Q_1 and Q_2 is:
  1. Along the line joining them
  2. Directly proportional to the product $Q_1 Q_2$ of the charges
  3. Inversely proportional to the square of distance R between them.

Expressed mathematically,

$$F = \frac{k Q_1 Q_2}{R^2} \tag{1}$$

where k is the proportionality constant whose value depend on the choice of system of units. In **SI** unit, charges Q_1 and Q_2 are in coulombs (C), the distance R is in meters (m), and the force F is in newtons (N) so that $k = 1/4\pi\varepsilon_0$. The constant ε_0 as the *permittivity of free space* (in farads per meter) and has the value
$\varepsilon_0 = 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi}$ F/m or $k = \frac{1}{4\pi\varepsilon_0}$ m/F

Coulomb's law and Field Intensity

$$F = \frac{kQ_1Q_2}{R^2} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2} \tag{2}$$

If point charges Q_1 and Q_2 are located at point having position vectors $\vec{r}_1$ and $\vec{r}_2$, then the force $\vec{F}_{12}$ on Q_2 due Q_1, is given by

$$\vec{F}_{12} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\widehat{a}_{R_{12}} \tag{3}$$

where

$$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1 \tag{4}$$

$$R = \mid \vec{R}_{12} \mid \tag{5}$$

$$\widehat{a}_{R_{12}} = \frac{\vec{R}_{12}}{R} \tag{6}$$

Then, we write eq.(3) as

$$\vec{F}_{12} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^3}\vec{R}_{12} \tag{7}$$

or

$$\vec{F}_{12} = \frac{Q_1Q_2(\vec{r}_2 - \vec{r}_1)}{4\pi\varepsilon_0 \mid \vec{r}_2 - \vec{r}_1 \mid^3} \tag{8}$$

Coulomb's law and Field Intensity

It is worthwhile to note that

- As shown, the force $\vec{F}_{21} = \mid \vec{F}_{12} \mid \hat{a}_{R_{21}} = \mid \vec{F}_{12} \mid (-\hat{a}_{R_{12}})$

  or $\vec{F}_{12} = -\vec{F}_{21}$

  since $\hat{a}_{R_{21}} = -\hat{a}_{R_{12}}$

- Like charges (charges of the same sign) repel each other, while unlike charge attract.

- The distance R between the charged bodies Q_1 and Q_2 must be large compared with the linear dimensions of the bodies; that is, Q_1 and Q_2 must be point charge.

- Q_1 and Q_2 must be static (at rest).

- The sign of Q_1 and Q_2 must be taken into account in eq.(3) for likes charges $Q_1 Q_2 > 0$. For unlike charges $Q_1 Q_2 < 0$.

Coulomb's law and Field Intensity

If we have more than two point charges, we can use the *principle of superposition* to determine the force on a particular charge. The principle state that if there are N charges $Q_1, Q_2, ..., Q_N$ located, respectively, at points with position vectors $\vec{r}_1, \vec{r}_2, ..., \vec{r}_N$ the resultant force $\vec{F}$ on a charge Q located at point $\vec{r}$ is the vector sum of the forces exerted on Q by each of the charge $Q_1, Q_2, ..., Q_N$. Hence:

$$\vec{F} = \frac{QQ_1(\vec{r} - \vec{r}_1)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_1 \mid^3} + \frac{QQ_2(\vec{r} - \vec{r}_2)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_2 \mid^3} + ... + \frac{QQ_N(\vec{r} - \vec{r}_N)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_N \mid^3} \tag{9}$$

or

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{Q_k(\vec{r} - \vec{r}_k)}{\mid \vec{r} - \vec{r}_k \mid^3} \tag{10}$$

Coulomb's law and Field Intensity

We can now introduce the concept of *electric field intensity*:

The **electric field intensity** (or **electric field strength**) $\vec{E}$ is the force per unit charge when placed in an electric field.
Thus

$$\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q} \tag{11}$$

or simply

$$\vec{E} = \frac{\vec{F}}{Q} \tag{12}$$

For $Q > 0$, the electric field intensity $\vec{E}$ is obviously in the direction of the force $\vec{F}$ and is measured in newtons per coulomb or volts per meter. The electric field intensity at point $\vec{r}$ due to a point charge located at $\vec{r'}$ is readily obtained from eqs.(3) and (12) as

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R = \frac{Q(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r'} \mid^3} \tag{13}$$

Coulomb's law and Field Intensity

For N point charges $Q_1, Q_2, ..., Q_N$ located at $\vec{r}_1, \vec{r}_2, ..., \vec{r}_N$, the electric field intensity at point $\vec{r}$ is obtained from eqs. (10) and (12) as

$$\vec{E} = \frac{Q_1(\vec{r} - \vec{r}_1)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_1 \mid^3} + \frac{Q_2(\vec{r} - \vec{r}_2)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_2 \mid^3} + ... + \frac{Q_N(\vec{r} - \vec{r}_N)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_N \mid^3} \tag{14}$$

or

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{Q_k(\vec{r} - \vec{r}_k)}{\mid \vec{r} - \vec{r}_k \mid^3} \tag{15}$$

Electric Field due to continuous charge distribution

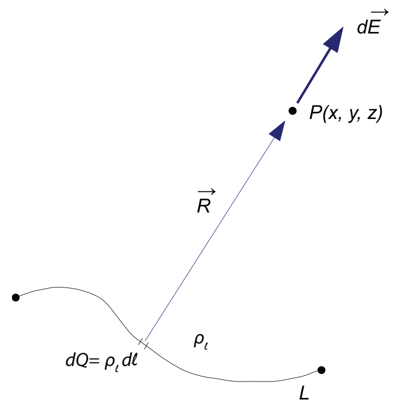
So far we have considered only forces and electric field due to point charges, which are essentially charges occupying very small physical space. It is also possible to have continuous charge distribution along line, surface, or in a volume.

It is customary to denote the line charge density, surface charge density, and volume charge density by ρ_ℓ (in C/m), ρ_s (in C/m^2), and ρ_v (in C/m^3). These must not be confused with ρ (without subscript), used for radial distance in cylindrical coordinates.

Electric Field due to continuous charge distribution

- For a linear charge density ρ_ℓ (C/m), the elemental charge $dQ = \rho_\ell \, d\ell$ and the differential field at a point P would be

$$d\vec{E} = \frac{\rho_\ell d\ell}{4\pi\varepsilon_0 R^2} \, \widehat{a}_R \tag{16}$$



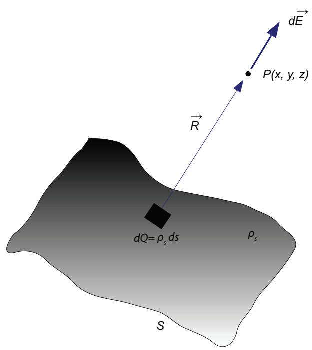
The total field at the observation point P is obtained by integrating over the line or curve L

$$\vec{E} = \int_L \frac{\rho_\ell \, \widehat{a}_R}{4\pi\varepsilon_0 R^2} d\ell \tag{17}$$

Electric Field due to continuous charge distribution

- For a surface charge density ρ_s (C/m^2), the elemental charge $dQ = \rho_s\, dS$ and the differential field at a point P would be

$$d\vec{E} = \frac{\rho_s ds}{4\pi\varepsilon_0 R^2}\, \widehat{a}_R \tag{18}$$



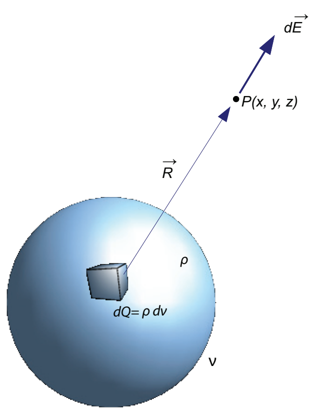
The total field at the observation point P is obtained by integrating over surface S

$$\vec{E} = \int_s \frac{\rho_s\, \widehat{a}_R}{4\pi\varepsilon_0 R^2} dS \tag{19}$$

Electric Field due to continuous charge distribution

- For a volume charge density ρ (C/m^3), the elemental charge $dQ = \rho_v\, dv$ and the differential field at the point P would be

$$d\vec{E} = \frac{\rho_v dv}{4\pi\varepsilon_0 R^2}\,\hat{a}_R \tag{20}$$



The total field at the observation point P is obtained by integrating over volume v

$$\vec{E} = \int_v \frac{\rho_v\,\hat{a}_R}{4\pi\varepsilon_0 R^2} dv \tag{21}$$

Problems

1. Point charges 1 mC and -2 mC are located at $(3, 2, -1)$ and $(-1, -1, 4)$ respectively. Calculate the electric force on a 10 nC charge located at $(0, 3, 1)$ and the electric field intensity at that point.

2. Point charges 5 nC and -2 nC are located $(2, 0, 4)$ and $(-3, 0, 5)$, respectively.
   (a) Determine the force on a 1 nC point charge located at $(1, -3, 7)$.
   (b) Find the electric field $\vec{E}$ at $(1, -3, 7)$.

3. A circular ring of radius a carries a uniform charge ρ_ℓ C/m and is placed on the xy-plane with axis the same as the z-axis.
   (a) Show that
   $$\vec{E}(0, 0, h) = \frac{\rho_\ell a h}{2\epsilon_0[h^2 + a^2]^{3/2}}\,\widehat{a}_z$$
   (b) What values of h gives the maximum value of $\vec{E}$?
   (c) If the total charge on the ring is Q, find $\vec{E}$ as $a \to 0$.

4. The finite sheet $0 \leq x \leq 1$, $0 \leq y \leq 1$ on the $z = 0$ plane has a charge density $\rho_s = xy(x^2 + y^2 + 25)^{3/2}$ nC/m^2. Find
   (a) The total charge on the sheet.
   (b) The electric field at $(0, 0, 5)$.
   (c) The force experimented by a -1 mC charge located at $(0, 0, 5)$.

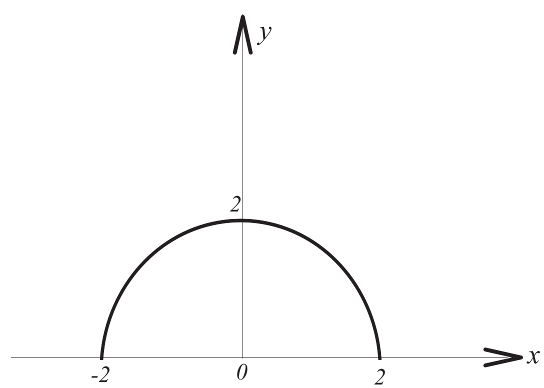
5. A square plate described by $-2 \leq x \leq 2$, $-2 \leq y \leq 2$, $z = 0$ carries a charge 12 $|\,y\,|$ mC/m^2. Find the total charge on the plate and the electric field intensity at $(0, 0, 10)$.

6. Planes $x = 2$ and $y = 3$, respectively, carry charges 10 nC/m^2 and 15 nC/m^2. If the line $x = 0$, $z = 2$ carries charge 10π nC/m, calculate $\vec{E}$ at $(1, 1, -1)$ due to the three charge distributions.

7. Planes $x = 2$ and $y = 3$, respectively, carry charges 10 nC/m^2 and 15 nC/m^2. If the line $x = 0$, $z = 2$ carries charge 10π nC/m is rotated through 90^0 about the point $(0, 2, 2)$ so that it become $x = 0$, $y = 2$, find $\vec{E}$ at $(1, 1, -1)$.

8. Charges $+Q$ and $+3Q$ separated by a distance 2 m. A third charge is located such that the electrostatic system is in equilibrium. Find the location and the value of the third charge in terms of Q.

9. A point charge Q is located at point $P(0, -4, 0)$, while a 10 nC charge is uniformly distributed along a circular ring as shown in the next Figure. Find the value of Q such that $E(0, 0, 0) = 0$.



10. (a) Show that the electric field at point $(0, 0, h)$ due to the rectangle described by $-a \leq x \leq a$, $-b \leq y \leq b$, $z = 0$ carrying uniform charge ρ_s C/m^2 is

$$\vec{E}(0, 0, h) = \frac{\rho_s}{\pi\epsilon_0} tan^{-1}[\frac{ab}{h(a^2 + b^2 + h^2)^{1/2}}]\, \hat{a}_z$$

(b) If $a = 2$, $b = 5$, $\rho_s = 10^{-5}$, find the total charge on the plate and the electric field intensity at $(0, 0, 10)$.

ELECTRIC FLUX DENSITY

Electric Field Intensity

The electric field intensity at point $\vec{r}$ due to a point charge located at $\vec{r'}$ is readily obtained from eq.(1),

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R = \frac{Q(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 \mid \vec{r}-\vec{r'} \mid^3} \tag{1}$$

For N point charges $Q_1, Q_2, ..., Q_N$ located at $\vec{r}_1, \vec{r}_2, ..., \vec{r}_N$, the electric field intensity at point $\vec{r}$ is obtained from eqs. (2) and (3) as

$$\vec{E} = \frac{Q_1(\vec{r}-\vec{r}_1)}{4\pi\varepsilon_0 \mid \vec{r}-\vec{r}_1 \mid^3} + \frac{Q_2(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0 \mid \vec{r}-\vec{r}_2 \mid^3} + ... + \frac{Q_N(\vec{r}-\vec{r}_N)}{4\pi\varepsilon_0 \mid \vec{r}-\vec{r}_N \mid^3} \tag{2}$$

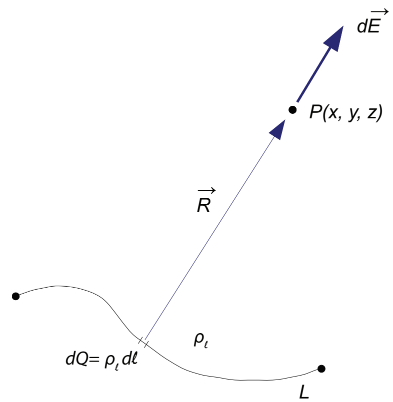
or

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k(\vec{r}-\vec{r}_k)}{\mid \vec{r}-\vec{r}_k \mid^3} \tag{3}$$

Electric Field Intensity

- For a linear charge density ρ_ℓ (C/m), the elemental charge $dQ = \rho_\ell\, d\ell$ and the differential field at a point P would be

$$d\vec{E} = \frac{\rho_\ell d\ell}{4\pi\varepsilon_0 R^2}\,\widehat{a}_R \tag{4}$$



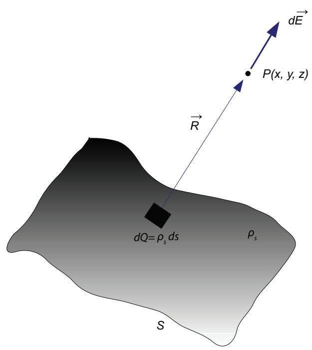
The total field at the observation point P is obtained by integrating over the line or curve L

$$\vec{E} = \int_L \frac{\rho_\ell\,\widehat{a}_R}{4\pi\varepsilon_0 R^2} d\ell \tag{5}$$

Electric Field Intensity

- For a surface charge density ρ_s (C/m^2), the elemental charge $dQ = \rho_s\, dS$ and the differential field at a point P would be

$$d\vec{E} = \frac{\rho_s ds}{4\pi\varepsilon_0 R^2}\,\widehat{a}_R \tag{6}$$



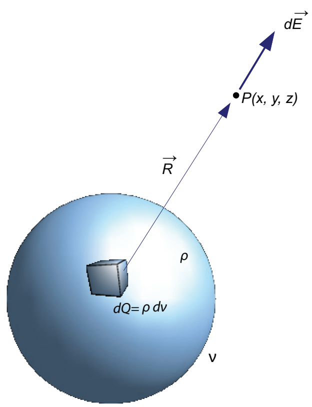
The total field at the observation point P is obtained by integrating over surface S

$$\vec{E} = \int_s \frac{\rho_s\,\widehat{a}_R}{4\pi\varepsilon_0 R^2} dS \tag{7}$$

Electric Field Intensity

- For a volume charge density ρ (C/m^3), the elemental charge $dQ = \rho_v \, dv$ and the differential field at the point P would be

$$d\vec{E} = \frac{\rho_V dV}{4\pi\varepsilon_0 R^2}\,\widehat{a}_R \tag{8}$$



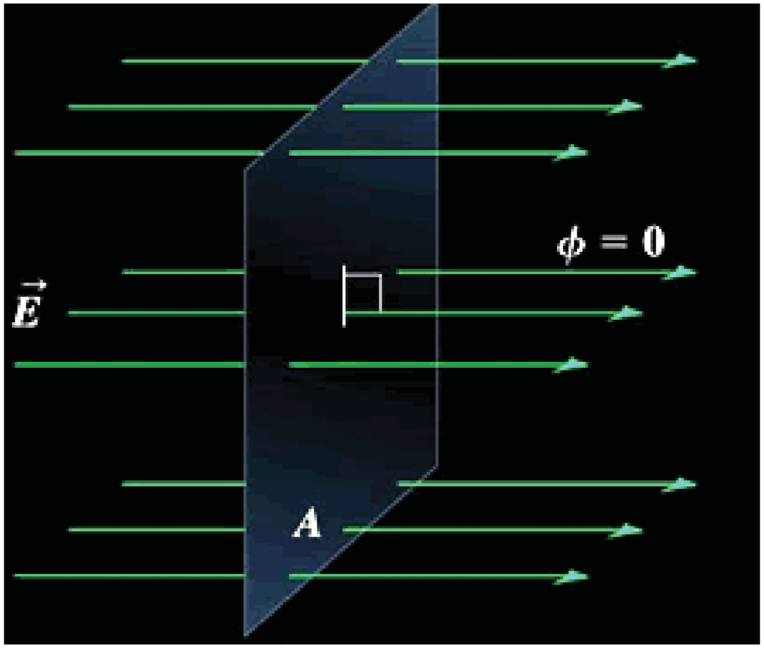
The total field at the observation point P is obtained by integrating over volume v

$$\vec{E} = \int_V \frac{\rho_V\,\widehat{a}_R}{4\pi\varepsilon_0 R^2} dV \tag{9}$$

Electric Flux Density

- Eqs. (1) to (9) show that the electric field intensity is dependent on the medium in with the charge is placed (free space in this case ε_0).

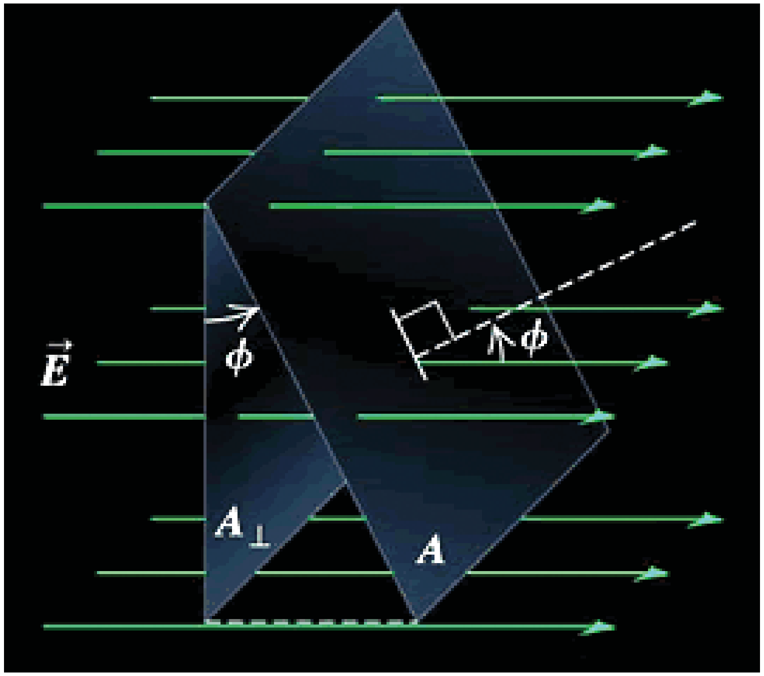
- Electric field $\vec{E}$ is perpendicular to area A; the angle between $\vec{E}$ and line perpendicular to the surface is zero.



- The flux is $\Phi_E = EA$.

Electric Flux Density

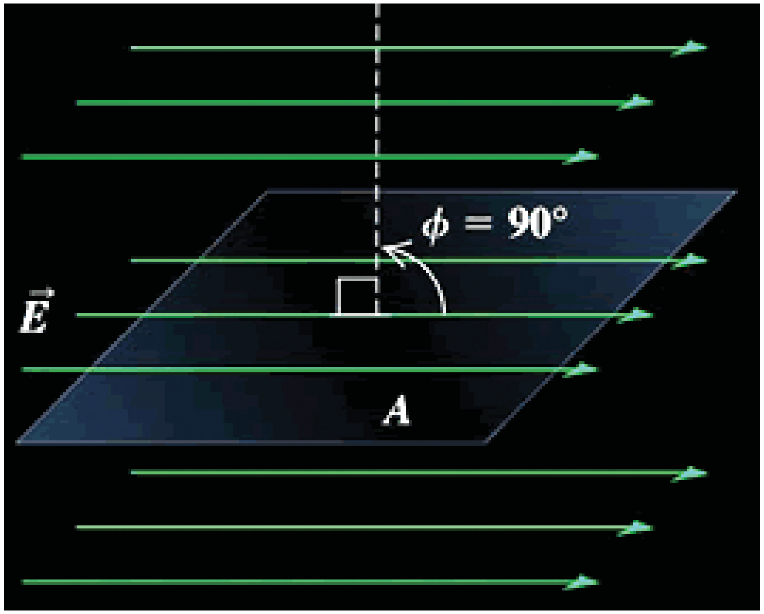
- Area A is tilted at an angle Φ from the perpendicular to $\vec{E}$.



- The flux is $\Phi_E = EA\cos(\Phi)$.

Electric Flux Density

- Area A is parallel to $\vec{E}$ (tilted at 90^0 from the perpendicular to $\vec{E}$).



- The flux is $\Phi_E = EA\cos(90^0) = 0$.

Electric Flux Density

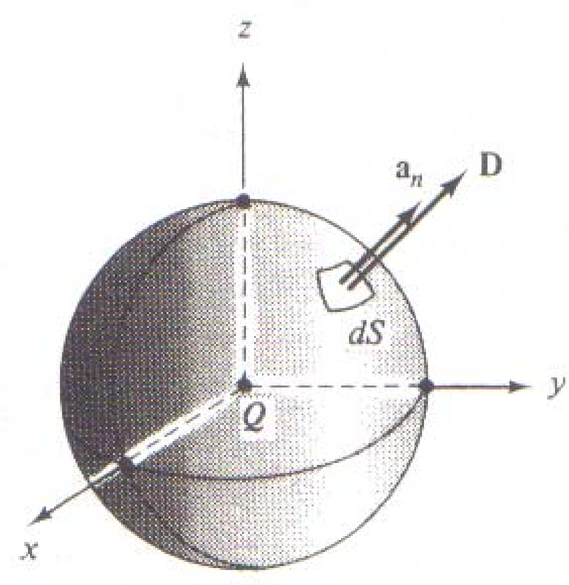
- Suppose a new vector field $\vec{D}$ is defined by

$$\vec{D} = \varepsilon_0 \vec{E} \tag{10}$$

- The electric flux Ψ in terms of $\vec{D}$ is

$$\Psi = \int_s \vec{D} \cdot d\vec{S} \tag{11}$$

- The electric flux is measured in coulombs. The vector field $\vec{D}$ is called the electric flux density and is measured in coulombs per square meter.



GAUSS'S LAW - MAXWELL'S EQUATION

Gauss's Law

- Gauss's law constitutes one of the fundamental laws of electromagnetism. **Gauss's law** state that the total electric flux Ψ through any *closed* surface is equal to the total charge enclosed by that surface

- Thus

$$\Psi = Q_{enc} \tag{12}$$

- That is,

$$\Psi = \oint_s d\Psi = \oint_s \vec{D} \cdot d\vec{S} =$$

$$\textit{total charge enclosed } Q = \int_V \rho_V dV \tag{13}$$

- or

$$Q = \oint_s \vec{D} \cdot d\vec{S} = \int_V \rho_V dV \tag{14}$$

Gauss's Law

- By applying divergence theorem to the middle term in eq (14), we have

$$\oint_s \vec{D} \cdot d\vec{S} = \int_V \nabla \cdot \vec{D} dV \tag{15}$$

- Comparing the two integrals results is

$$\rho_V = \nabla \cdot \vec{D} \tag{16}$$

- Eq (16) is the first of the four *Maxwell's* equations to be derived.
- This equation states that the volume charge density is the same as the divergence of the electric flux density.
- Equations (14) and (16) are basically stating *Gauss's law* in different ways; eq. (14) is the integral form, whereas eq. (16) is the differential or point form of Gauss's law.
- Gauss's law provides an easy means of finding $\vec{E}$ or $\vec{D}$ for symmetrical charge distributions such as a point charge, an infinite line charge, an infinite cylindrical surface charge, and a spherical distribution of charge.

Gauss's Law

- The *divergence of a static field* is used to determine when a region has *sources* (*net positive charge*) or *sinks* (*net negative charge*). By definition, the divergence of the electric lux density at a point P is

$$\text{div}\vec{D} = \nabla \cdot \vec{D} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{D} \cdot d\vec{S}}{\Delta V} = \lim_{\Delta V \to 0} \frac{Q_{enclosed}}{\Delta V} = \rho \tag{17}$$

where S is the boundary of ΔV.

- For a general vector $\vec{A}$, the definitions for the divergence in the three coordinate systems of interes are:

$$\text{Cartesian :} \qquad \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{18}$$

$$\text{Cylindrical :} \qquad \nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{19}$$

$$\text{Spherical :} \qquad \nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \tag{20}$$

APPLICATIONS OF GAUSS'S LAW

Gauss's Law

- For applying Gauss's law to calculate the electric field involves first knowing whether symmetry exist.

- Once it has been found that symmetric charge distribution exit, we construct a mathematical closed surface (known as Gaussian surface).

- The surface is chosen such that $\vec{D}$ is normal or tangential to the Gaussian surface.

- When $\vec{D}$ is normal to the surface, $\vec{D} \cdot d\vec{S} = DdS$ because $\vec{D}$ is constant on the surface.

- When $\vec{D}$ is tangential to the surface, $\vec{D} \cdot d\vec{S} = 0$.

- Thus we must choose a surface that has some of the symmetry exhibited by the charge distribution.

Problems

NOTE: The integral and point forms of Gauss' law are related by the *divergence theorem* given by

$$\Psi = \oint_S \vec{D} \cdot d\vec{S} = \int_V (\nabla \cdot \vec{D}) dV = \int_V \rho_V dV = Q_{\text{enclosed}} \tag{21}$$

where S is the closed surface boundary of the volume V.

1. **Example:** In the region $0 < r < 1$ m, $\vec{D} = (-2\text{x}10^{-4}/r)\,\widehat{a}_r$ (C/m^2) and for $r > 1$ m, $\vec{D} = (-4\text{x}10^{-4}/r^2)\,\widehat{a}_r$ (C/m^2) in spherical coordinates. Find the charge density in both region.

2. Charge in the form of a plane sheet with density $\rho_s = 40$ (μC/m^2) is located at $z = -0.5$ (m). A uniform line charge of $\rho_\ell = -6$ (μC/m) lies along the $y-$axis. What net flux crosses the surface of a cube 2 (m) on an edge, centered at the origin of coordinate system?

3. Determine the flux crossing a 1 mm by 1 mm area on surface of a cylindrical shell at $r = 10$ m, $z = 2$ m, $\psi = 53.3^o$ if $\vec{D} = 2x\,\widehat{a}_x + 2(1-y)\,\widehat{a}_y + 4z\,\widehat{a}_z$.

ELECTRIC POTENTIAL

Electric Potential

- We can obtain the electric file intensity $\vec{E}$ due to a charge distribution from Coulomb's law in general or, when the charge distribution is symmetric, from Gauss's law.
- Another way of obtaining $\vec{E}$ is from the electric scalar potential V.
- In a sense, this way of finding $\vec{E}$ is easier because it is easier to handle scalars than vectors.
- Suppose we wish to move a point charge Q from point A to point B in a electric field $\vec{E}$ as show in Figure 1.

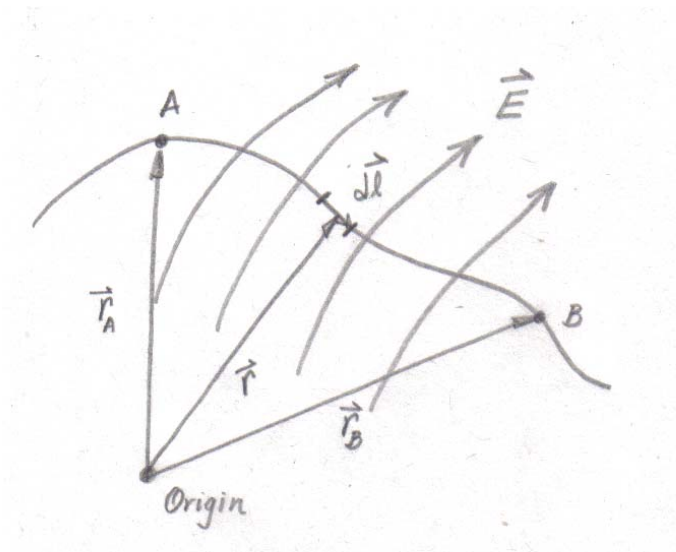


Figure 1 Displacement of point charge Q in an electric field $\vec{E}$.

Electric Potential

- From Coulomb's law, the force on Q is $\vec{F} = Q\vec{E}$ so that the *work done* in displacing the charge by $d\vec{\ell}$ is

$$dW = -\vec{F} \cdot d\vec{\ell} = -Q\vec{E} \cdot d\vec{\ell} \tag{1}$$

The negative sing indicate that the work is being done by external agent.

- Thus the total work done, or the potential energy required, in moving Q from A to B, is

$$W = -Q \int_A^B \vec{E} \cdot d\vec{\ell} \tag{2}$$

- Dividing W by Q in eq. (2) gives the potential energy per unit charge. This quantity, denoted by V_{AB}, is known as the *potential difference* between points A and B. Thus

$$V_{AB} = \frac{W}{Q} = -\int_A^B \vec{E} \cdot d\vec{\ell} \tag{3}$$

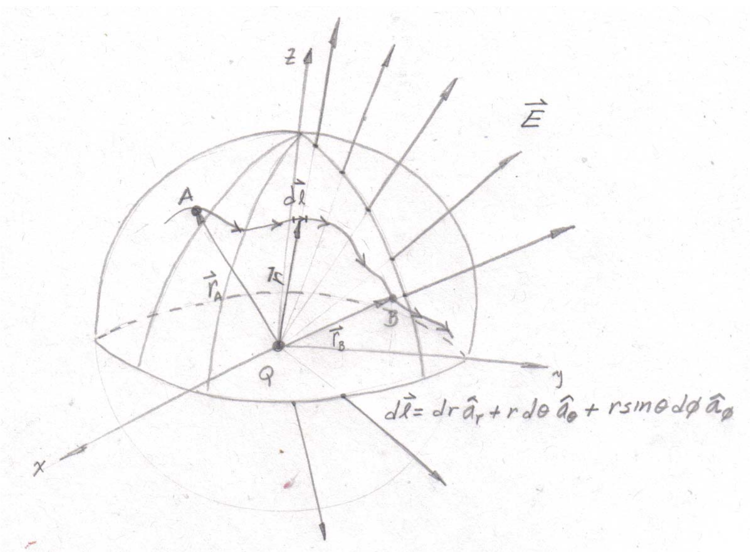
Electric Potential

Note that:

- In determining V_{AB}, A is the initial point while B is the final point.
- If V_{AB} is negative, there is a loss in potential energy in moving Q from A to B; this implies that the work is being done by the field.
- If V_{AB} is positive, there is a gain in potential energy in the movement; an external agent performs the work.
- V_{AB} is independent of the path taken (to be shown a little later).
- V_{AB} is measured in joules per meter, commonly referred to a volts (V).

Electric Potential

As an example, if the $\vec{E}$ field in Figure 1 is due to a point charge Q located at the origin, then



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \tag{4}$$

so eq.(3)

$$V_{AB} = -\int_A^B \vec{E}\cdot d\vec{\ell} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \cdot dr\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}[\frac{1}{r_B} - \frac{1}{r_A}] \tag{5}$$

Electric Potential

or

$$V_{AB} = V_B - V_A \tag{6}$$

where V_B and V_A are the *potentials* (or *absolute potentials*) at B and A, respectively.

- Thus the potential difference V_{AB} may be regarded as the potential at B with reference to A.

- Note from eq. (5) that because $\vec{E}$ points in the radial direction, any contribution from displacement in the θ or ϕ direction is wiped out by the dot product $\vec{E} \cdot d\vec{\ell} = E \cos\alpha d\ell = E dr$, where α is the angle between $\vec{E}$ and $d\vec{\ell}$. Hence the potential difference V_{AB} is independent of the path as asserted earlier.

- In general, vectors whose line integral does not depend on the path of integration are called conservative. This $\vec{E}$ is conservative.

Electric Potential

- The potential at any point ($r_B \to r$) due to charge Q located at the origin is
$$V = \frac{Q}{4\pi\varepsilon_0 r} \tag{7}$$

- The potential at any point is the potential difference between that point and a chosen point (or reference point) at which the potential is zero.

- In other words, by assuming zero potential at infinity, the potential at a distance r from the point charge is the work done per unit cherge by an external agent in transferring a test charge from infinity to that point. Thus
$$V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{\ell} \tag{8}$$

Electric Potential

- If the point charge Q in eq. (7) is not located at the origin but at a point whose position vector is $\vec{r}'$, the potential $V(x, y, z)$ or simply $V(\vec{r})$ at $\vec{r}$ becomes

$$V = \frac{Q}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}' \mid} \tag{9}$$

- The superposition principle, wich we applied to electric field, applies to potential. For n point charges $Q_1, Q_2, ..., Q_n$ located at points with position vectors $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$, the potential at $\vec{r}$ is

$$V(\vec{r}) = \frac{Q_1}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_1 \mid} + \frac{Q_2}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_2 \mid} + ... + \frac{Q_n}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_n \mid} \tag{10}$$

or

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{Q_k}{\mid \vec{r} - \vec{r}_k \mid} \quad \text{point charges} \tag{11}$$

Electric Potential

- For continuous charge distribution, we replace Q in eq. (11) with charge element $\rho_\ell \, d\ell$, $\rho_S \, dS$, or $\rho_V \, dV$ and the summation becomes an integration, so the potential at $\vec{r}$ become

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\rho_\ell(\vec{r'}) \, d\ell'}{|\, \vec{r} - \vec{r'} \,|} \quad \text{line charge} \tag{12}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_S(\vec{r'}) \, dS'}{|\, \vec{r} - \vec{r'} \,|} \quad \text{surface charge} \tag{13}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_V(\vec{r'}) \, dV'}{|\, \vec{r} - \vec{r'} \,|} \quad \text{volume charge} \tag{14}$$

where the primed coordinates are used customarily to denote source point location and the unprimed coordinates refer to field point (the point at which V is to be determined).

Electric Potential

The following points should be noted:

- We recall that in obtaining eqs. (7) to (14), the zero potential (reference) point has been chosen arbitrarily to be at infinity. If any other point is chosen as reference, eq.(9), for example, becomes

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C \tag{15}$$

  where C is a constant that is determined at the chosen point of reference. The same idea applies to (9) to (14).

- The potential at a point can be determined in two ways depending on whether the charge distribution or $\vec{E}$ is known. If the charge distribution is known, we use on of the (7) to (14) depending on the charge distribution.

- If $\vec{E}$ is known, we simply use

$$V = -\int \vec{E} \cdot d\vec{\ell} + C \tag{16}$$

- The potential difference V_{AB} can be found generally from

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{\ell} = \frac{W}{Q} \tag{17}$$

RELATIONSHIP BETWEEN $\vec{E}$ and V

Relationship between $\vec{E}$ and V

- The potential difference between points A and B is independent of the path taken. Hence

$$V_{BA} = -V_{AB}$$

that is, $V_{BA} + V_{AB} = \oint \vec{E} \cdot d\vec{\ell} = 0$ or

$$\oint_L \vec{E} \cdot d\vec{\ell} = 0 \tag{18}$$

This shows that the line integral of $\vec{E}$ along a closed path as shown in Figure 3, must be zero.

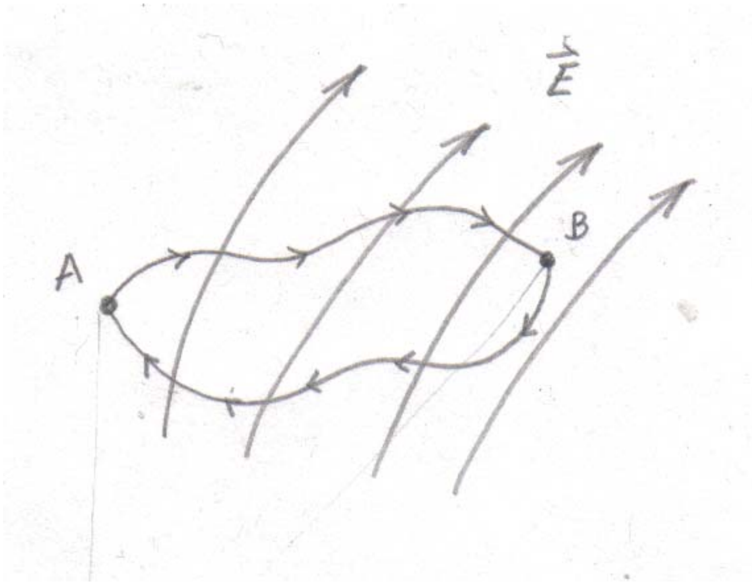


Figure 3 The conservative nature of an electric field.

Relationship between $\vec{E}$ and V

- Physically, this implies that no net work is done in moving a charge along a closed path in an electrostatic field.

- Applying Stokes's theorem to eq.(18) gives

$$\oint_L \vec{E} \cdot d\vec{\ell} = \int_s (\nabla \times \vec{E}) \cdot d\vec{S} = 0 \tag{19}$$

  or

$$\nabla \times \vec{E} = 0 \tag{20}$$

- Any vector field that satisfies eq.(19) or eq.(20) is said to be conservative, or irrotational.

- In other words, vectors whose line integral does not depend on the path of integration are called conservative vectors.

- Thus an electrostatic field is a conservative field. Equations (19) or (20) is referred as *Maxwell's equation* (the second Maxwell equation to be derived) for static electric field.

Relationship between $\vec{E}$ and V

- Equation (19) is the integral form, and eq. (20) is the differential form; they both depict the conservative nature of and electrostatic field.

- From the way we defined potential, $V = -\int \vec{E} \cdot d\vec{\ell}$, it follow that

$$dV = -\vec{E} \cdot d\vec{\ell} = -E_x dx - E_y dy - E_z dz \tag{21}$$

- But from calculus of multivariables, a total charge in $V(x, y, z)$ is the sum of partial charges with respect to x, y, z variables:

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \tag{22}$$

- Comparing eqs. (21) and (22)

$$\vec{E} = -\nabla V \tag{23}$$

  that is, the electric field is the gradient of V.

Relationship between $\vec{E}$ and V

- The negative sign shows that the direction of $\vec{E}$ is opposite to the direction in which V increases; $\vec{E}$ is directed from higher to lower levels of V.
- Equation (23) shows another way to obtain $\vec{E}$ field apart from using Coulomb's or Gauss's law.
- One may wonder how one function V can possibly contain all the information that the three components of $\vec{E}$ carry.

Problems

1. An electrostatic field is given by $\vec{E} = \frac{x}{2} + 2y\,\widehat{a}_x + 2x\,\widehat{a}_y$. Find the work done in moving a point charge $Q = -20\mu$C
   (a) From the origin to $(4, 0, 0)$ m.
   (b) From $(4, 0, 0)$ m to $(4, 2, 0)$ m.
   (c) From $(4, 2, 0)$ m to $(0, 0, 0)$ m.

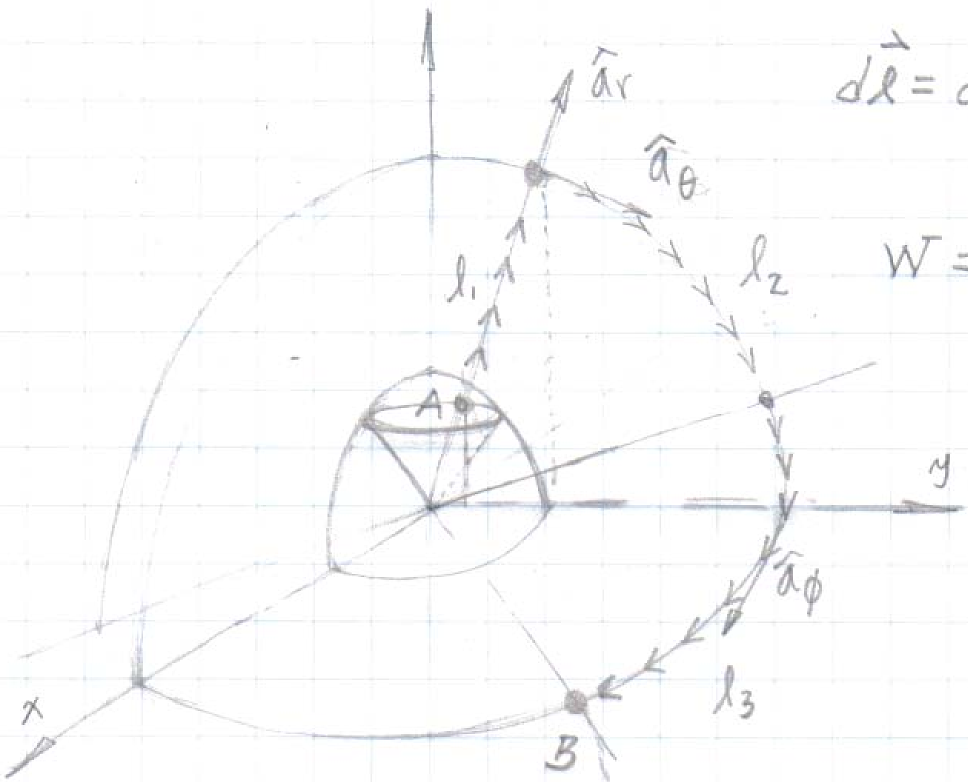
2. Two point charges -4μC and 5μC are located at $(2, -1, 3)$ and $(0, 4, -2)$, respectively. Find the potential at $(1, 0, 1)$.

3. A total charge of $40/3$ nC is uniformly distributed in the form of a circular disk of radius 2m. find the potential due to the charge at point of the axis, 2m from the disk.

4. A point charge of 5 nC is located at $(-3, 4, 0)$, while line $y = 1$, $z = 1$ carries uniform charge 2 nC/m.
   (a) If $V = 0$ V at $O(0, 0, 0)$, find V at $A(5, 0, 1)$.
   (b) If $V = 100$ V at $B(1, 2, 1)$, find V at $C(-2, 5, 3)$.
   (c) If $V = -5$ V at O, find V_{BC}.

Problems

1. Given the potential $V = \frac{10}{r^3}\sin\theta\cos\phi$,
   (a) Find the electric flux density $\vec{D}$ at $(2, \pi/2, 0)$.
   (b) Calculate de work done in moving a 10 μC charge from point $A(1, 30^0, 120^0)$ to $B(4, 90^0, 60^0)$.



$$d\vec{l} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$$

$$W = -Q\int_A^B \vec{E}\cdot d\vec{l}$$

$$\vec{E} = -\nabla V$$
$$= -\frac{10}{r^3}\left[(-2\sin\theta\cos\phi)\,\hat{a}_r + (\cos\theta\cos\phi)\,\hat{a}_\theta - \sin\phi\,\hat{a}_\phi\right]$$

Energy Density in Electrostatic Fields

Energy Density in Electrostatic Fields

- To determine the energy present in a assembly of charges, we must first determine the amount of work necessary to assemble them.

- Suppose we wish to position three point charge Q_1, Q_2 and Q_3 in an initially empty space shown shaded in Figure 1.

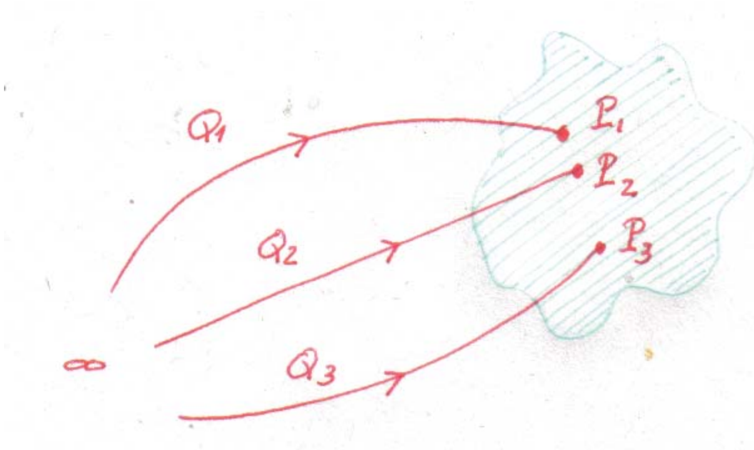


Figure 1 Assembling of charges.

- No work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field.

$$W = -Q_1 \int_A^B \vec{E} \cdot d\vec{\ell} = 0 \tag{1}$$

Energy Density in Electrostatic Fields

- The work to transferring Q_2 from infinity to P_2 is equal to the product of Q_2 and the potential V_{21} at P_2 due to Q_1.

- Similarly, the work done in positioning Q_3 at P_3 is equal to $Q_3(V_{32} + V_{31})$, where V_{32} and V_{31} are the potentials at P_3 due Q_2 and Q_1, respectively.

- Hence the total work done in positioning the three charges is

$$\begin{aligned} W_E &= W_1 + W_2 + W_3 \\ &= 0 + Q_2V_{21} + Q_3(V_{31} + V_{32}) \end{aligned} \tag{2}$$

- If the charges were positioned in reverse order,

$$\begin{aligned} W_E &= W_3 + W_2 + W_1 \\ &= 0 + Q_2V_{23} + Q_1(V_{12} + V_{13}) \end{aligned} \tag{3}$$

  where V_{23} is the potential at P_2 due to Q_3, V_{12} and V_{13} are, respectively, the potential at P_1 due to Q_2 and Q_3.

Energy Density in Electrostatic Fields

- Adding eqs. (2) and (3) gives

$$\begin{aligned} 2W_E &= Q_1(V_{12}+V_{13}) + Q_2(V_{21}+V_{23}) + Q_3(V_{31}+V_{32}) \\ &= Q_1V_1 + Q_2V_2 + Q_3V_3 \end{aligned} \tag{4}$$

- or

$$W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3) \tag{5}$$

where V_1, V_2 and V_3 are the total potentials at P_1,P_2 and P_3, respectively.

- In general, if there are n point charges, eq.(5) become

$$W_E = \frac{1}{2}\sum_{k=1}^{n} Q_kV_k \quad \text{(in joules)} \tag{6}$$

Energy Density in Electrostatic Fields

- If, instead of point charges, the region has a continuous charge distribution, the summation in eq.(6) becomes integration; that is,

$$W_E = \frac{1}{2}\int_L \rho_\ell V \, d\ell \quad \text{(line chrage)} \tag{7}$$

$$W_E = \frac{1}{2}\int_S \rho_S V \, dS \quad \text{surface chrage} \tag{8}$$

$$W_E = \frac{1}{2}\int_\nu \rho_\nu V \, d\nu \quad \text{volume chrage} \tag{9}$$

- Since $\rho_v = \nabla \cdot \vec{D}$, eq.(9) can be further developed to yield

$$W_E = \frac{1}{2}\int_\nu (\nabla \cdot \vec{D}) V \, d\nu \quad \text{volume chrage} \tag{10}$$

- But for any vector $\vec{A}$ and escalar V, the identity $\nabla \cdot V\vec{A} = \vec{A} \cdot \nabla V + V(\nabla \cdot \vec{A})$ or

$$(\nabla \cdot \vec{A})V = \nabla \cdot V\vec{A} - \vec{A} \cdot \nabla V \tag{11}$$

holds.

Energy Density in Electrostatic Fields

- Applying the identity in eq.(11) to (10), we get

$$W_E = \frac{1}{2}\int_{\nu}(\nabla\cdot\vec{D})V\,d\nu = \frac{1}{2}\int_{\nu}(\nabla\cdot V\vec{D})\,d\nu - \frac{1}{2}\int_{\nu}(\vec{D}\cdot\nabla V)\,d\nu \tag{12}$$

- But applying divergence theorem to the first term on the right-hand side of this equation, we have

$$W_E = \frac{1}{2}\oint_{S}(V\vec{D})\cdot d\vec{S} - \frac{1}{2}\int_{\nu}(\vec{D}\cdot\nabla V)\,d\nu \tag{13}$$

- Hence, eq.(13) reduce to

$$W_E = -\frac{1}{2}\int_{\nu}(\vec{D}\cdot\nabla V)\,d\nu = \frac{1}{2}\int_{\nu}(\vec{D}\cdot\vec{E})\,d\nu \tag{14}$$

- and since $\vec{E} = -\nabla V$ and $\vec{D} = \varepsilon_0\vec{E}$

$$W_E = \frac{1}{2}\int_{\nu}(\vec{D}\cdot\vec{E})\,d\nu = \frac{1}{2}\int_{\nu}\varepsilon_0 E^2\,d\nu \tag{15}$$

Energy Density in Electrostatic Fields

- From this, we can define electrostatic energy density w_E (in J/m^3) as

$$w_E = \frac{dW_E}{d\nu} = \frac{1}{2}\vec{D} \cdot \vec{E} = \frac{1}{2}\varepsilon_0 E^2 = \frac{D^2}{2\varepsilon_0} \tag{16}$$

- so eq.(14) may be written as

$$W_E = \int_\nu w_E d\nu \tag{17}$$

MAGNETOSTATICS

Magnetostatic

- In the last sections, we limited our discussions to static electric fields characterized by $\vec{E}$ or $\vec{D}$.

- We now focus our attention on static magnetic fields, which are characterized by $\vec{H}$ or $\vec{B}$.

- There are similarities and dissimilarities between electric and magnetic fields.

- As $\vec{E}$ and $\vec{D}$ are related according to $\vec{D} = \varepsilon\vec{E}$ for linear, isotropic material space, $\vec{H}$ and $\vec{B}$ are realted according to $\vec{B} = \mu\vec{H}$.

Magnetostatic

- Analogy between Electric ($\vec{D} = \varepsilon\vec{E}$) and Magnetic Fields ($\vec{B} = \mu\vec{H}$).

Term	Electric	Magnetic
Basic laws	$\vec{F} = \frac{Q_1 Q_2}{4\pi\varepsilon r^2}\,\hat{a}_r$	$d\vec{B} = \frac{\mu I d\vec{\ell} \times \hat{a}_r}{4\pi R^2}$
	$\oint \vec{D} \cdot d\vec{S} = Q_{enc}$	$\oint \vec{H} \cdot d\vec{\ell} = I_{enc}$
Force law	$\vec{F} = Q\vec{E}$	$\vec{F} = Q\vec{u} \times \vec{B}$
Source element	dQ	$Q\vec{u} = Id\vec{\ell}$
Field intensity	$E = \frac{V}{\ell}$ (V/m)	$H = \frac{I}{\ell}$ (A/m)
Flux density	$\vec{D} = \frac{\psi}{S}$ (C/m^2)	$\vec{B} = \frac{\psi}{S}$ (Wb/m^2)
Relationships between fields	$\vec{D} = \varepsilon\vec{E}$	$\vec{B} = \mu\vec{H}$
Potentials	$\vec{E} = -\nabla V$	$\vec{H} = -\nabla V_m$ ($\vec{J} = 0$)
	$V = \int \frac{\rho_\ell\, d\ell}{4\pi\varepsilon r}$	$A = \int \frac{\mu I\, d\ell}{4\pi R}$
Flux	$\psi = \int \vec{D} \cdot d\vec{S}$	$\psi = \int \vec{B} \cdot d\vec{S}$
	$\psi = Q = CV$	$\psi = LI$
	$I = C\frac{dV}{dt}$	$V = L\frac{dI}{dt}$
Energy density	$W_E = \frac{1}{2}\vec{D} \cdot \vec{E}$	$W_m = \frac{1}{2}\vec{B} \cdot \vec{H}$
Poisson's equation	$\nabla^2 V = \frac{\rho_V}{\varepsilon}$	$\nabla^2 A = -\mu\vec{J}$

Magnetostatic

- A definite link between electric and magnetic fields was established by Hans Christian Oersted (1777-1851) in 1820, Danish professor of physics.

- As we have noticed, an electrostatic field is produced by static or stationary charges.

- If the charges are moving with constant velocity, a static magnetic (or magnetostatic) field is produced.

- A magnetostatic field is produced by a constant current flow (or direct current).

- This current flow may be due to magnetization currents as in permanent magnets, electron-beam currents as in vacuum tubes, or condition currents as in current-carrying wires.

- In this section, we consider magnetic fields in free space due to direct current. Motors, transformers, microphones, compasses, telephone bell ringers, television focussing controls, advertising displays, magnetically levitated high-speed vehicles, memory store, magnetic separators, and so on, which play an important role in our everyday life, could not have been developed without an understanding of magnetic phenomena.

Magnetostatic

- There are two major laws governing magnetostatic fields: (1) Biot-Savart's law, and (2) Ampère's law.

- Like Coulomb's law, Biot-Savart's law is the general law of magnetostatics.

- Just as Gauss's law is a special case of Coulomb's law, Ampère's law is a special case of Biot-Savart's law and is easily applied in problems involving symmetrical current distribution.

Biot-Savart's Law

- **Biot-Savart law** states that the differential magnetic field intensity dH produced at a point P, as shown in Figure 1, by the differential current element $Id\ell$ is proportional to de product $Id\ell$ and the sine of the angle α between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element.

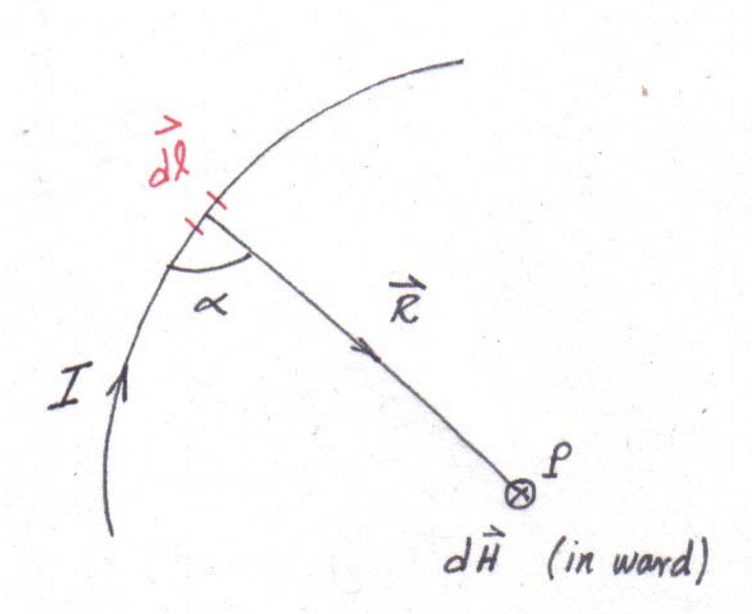


Figure 1 Magnetic field $d\vec{H}$ at P due to current element $Id\vec{\ell}$.

- That is,

$$dH = \frac{Id\ell \sin\alpha}{4\pi R^2} \tag{1}$$

Biot-Savart's Law

- From the definition of cross product, it is easy to notice that eq.(1) is better put in vector form as

$$d\vec{H} = \frac{Id\vec{\ell} \times \hat{a}_R}{4\pi R^2} = \frac{Id\vec{\ell} \times \vec{R}}{4\pi R^3} \tag{2}$$

  where $R = \mid \vec{R} \mid$ and $\hat{a}_R = \vec{R}/R$; $\vec{R}$ and $d\vec{\ell}$ are illustrated in Figure 1.

- The direction of $d\vec{H}$ can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current and the right-hand fingers encircling the wire in the direction of $d\vec{H}$ as shown in Figure 2.

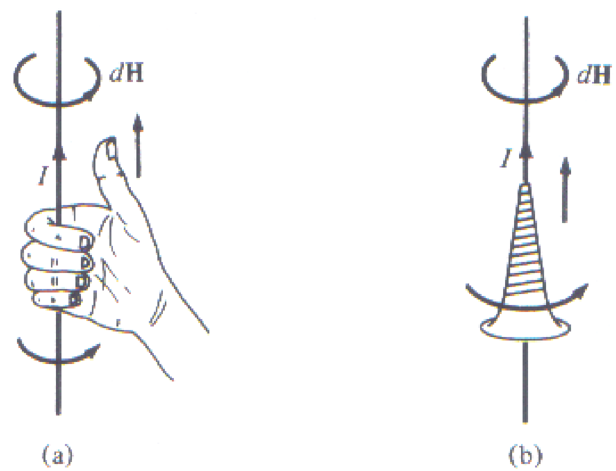


Figure 2 Determining the direction of $d\vec{H}$ using (a) the right-hand rule or (b) the right-handed-screw rule.

Biot-Savart's Law

- Just as we can have different charge configurations, we can have different current distributions: line current, surface current, and volume current as shown in Figure 3.

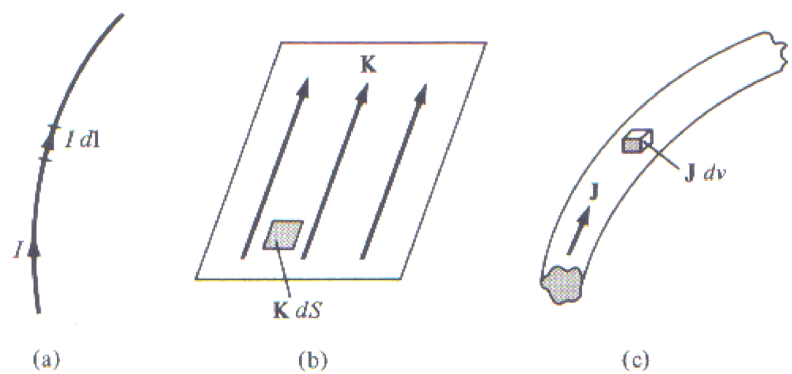


Figure 3 Current distributions: (a) line current, (b) surface current, (c) volume current.

- If we define $\vec{K}$ as the surface current density in A/m and $\vec{J}$ as the volume current density in A/m^2, the source element are related as

$$I d\vec{\ell} = \vec{K} dS = \vec{J} dV \tag{3}$$

Biot-Savart's Law

- Thus in terms of the distributed current source, the Biot-Savart law as in eq.(2) become

$$\vec{H} = \int_L \frac{I d\vec{\ell} \times \widehat{a}_R}{4\pi R^2} \quad \text{line current} \tag{4}$$

$$\vec{H} = \int_S \frac{\vec{K} dS \times \widehat{a}_R}{4\pi R^2} \quad \text{surface current} \tag{5}$$

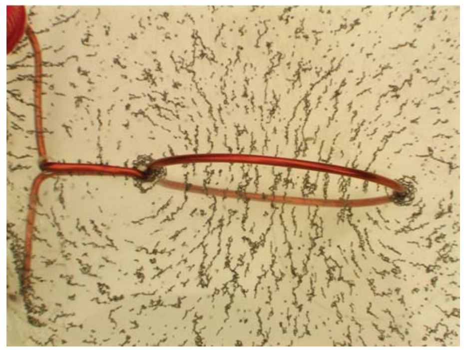
$$\vec{H} = \int_V \frac{\vec{J} dV \times \widehat{a}_R}{4\pi R^2} \quad \text{volume current} \tag{6}$$

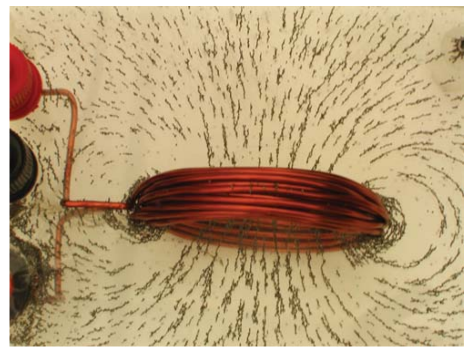
where $\widehat{a}_R$ is a unit vector pointing from the differential element of current to the point of interes.

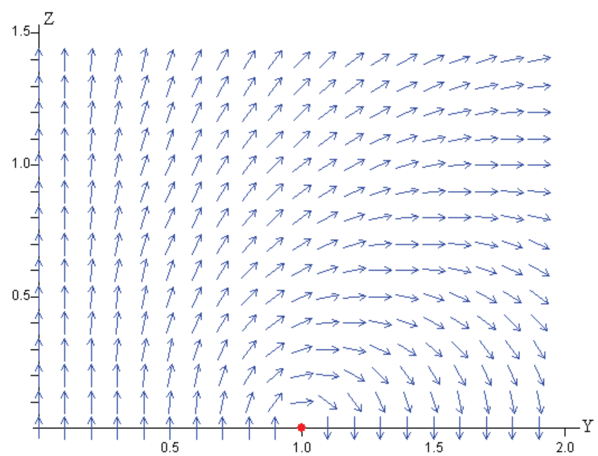
Biot-Savart's Law

- Circular lop of radius ρ, carries a direct current I

$$\vec{H} = \frac{I\rho^2}{2(\rho^2 + z^2)^{3/2}}\,\hat{a}_z \tag{7}$$





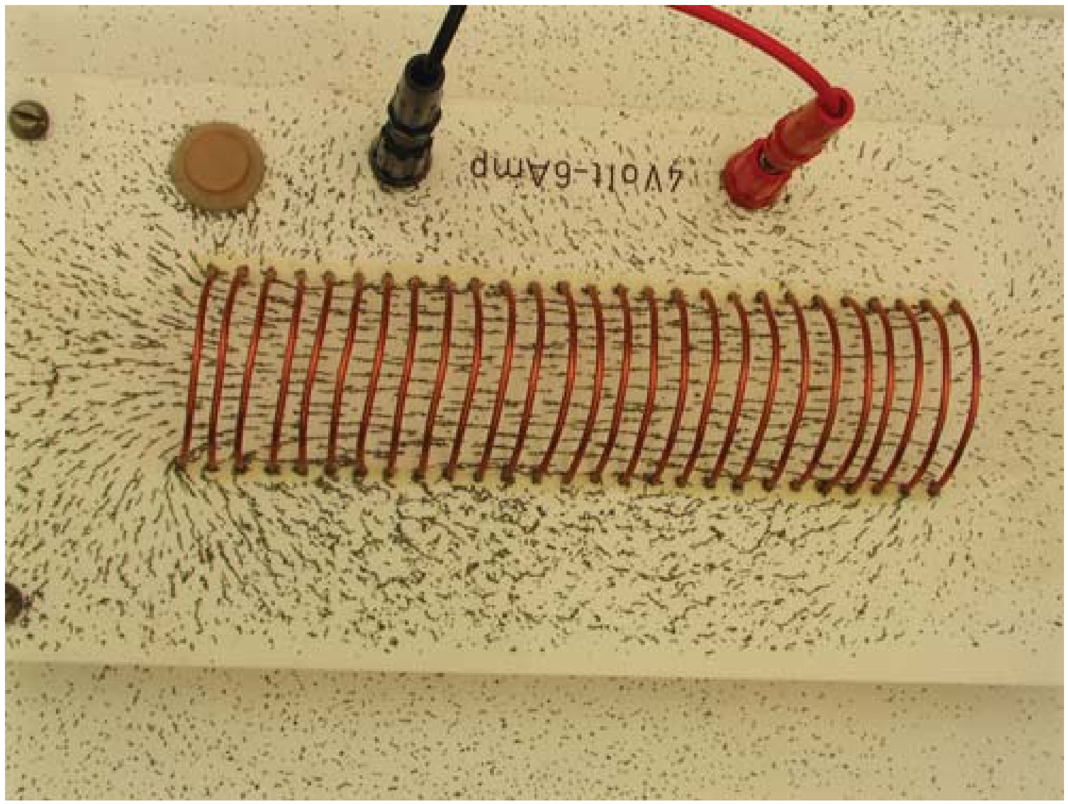


$$z = 0, \quad \vec{H} = \frac{I}{2\rho}\,\hat{a}_z \tag{8}$$

Biot-Savart's Law

- A solenoid of length ℓ, radius a and N turns of wire carries current I.

$$\vec{H} = \frac{IN}{2\sqrt{a^2 + \frac{\ell^2}{4}}}\,\hat{a}_z \tag{9}$$



$$\ell >> a, \quad \vec{H} = \frac{IN}{\ell}\,\hat{a}_z \tag{10}$$

AMPÈRE'S CIRCUIT LAW

Ampère's circuit law

- States that the integral of $\vec{H}$ around a *closed* path is the same as the net current I_{enc} enclosed by the path.

- In other words, the circulation of $\vec{H}$ equals I_{enc}: that is,

$$\oint \vec{H} \cdot d\vec{\ell} = I_{enc} \tag{1}$$

- Ampèr's law is similar to Gauss's law, since Ampèr's law is easily applied to determine $\vec{H}$ when the current distribution is symmetrical.

- By applying Stoke's theorem to the left-hand side of eq. (1), we obtain

$$I_{enc} = \oint_{\zeta} \vec{H} \cdot d\vec{\ell} = \int_{S} (\nabla \times \vec{H}) \cdot d\vec{S} \tag{2}$$

Ampère's circuit law

- But

$$I_{enc} = \int_S \vec{J} \cdot d\vec{S}, \tag{3}$$

where $\vec{J}$ is *current density*.

- Comparing the surface integrals in eqs. (2) and (3) clearly reveals that

$$\nabla \times \vec{H} = \vec{J} \tag{4}$$

- This is the third Maxwell's equation to be derived; it is essentially Ampère's law in differential (or point) form, whereas eq. (1) is the integral form.

- From eq. (4), we should observe that $\nabla \times \vec{H} = \vec{J} \neq 0$; that is a magnetostatic field is not conservative.

MAGNETIC FLUX DENSITY

Magnetic Flux Density

- The magnetic flux density $\vec{B}$ is similar to the electric flux density $\vec{D}$.

- As $\vec{D} = \varepsilon_o \vec{E}$ in free space, the magnetics flux density $\vec{B}$ is related to the magnetic field intensity $\vec{H}$ according to

$$\vec{B} = \mu_o \vec{H} \tag{1}$$

where μ_o is a constant known as the *permeability of free space*. The constant is in henrys por meter (H/m) and has the value of

$$\mu_o = 4\pi \times 10^{-7} \text{ H/m} \tag{2}$$

- The magnetic flux through a surface S is given by

$$\psi = \int_S \vec{B} \cdot d\vec{S} \tag{3}$$

where the magnetic flux ψ is in Webers (Wb) and the magnetic flux density is in Webers per square meter (Wb/m^2) or Tesla (T).

Magnetic Flux Density

- $$\psi = \int_S \vec{B} \cdot d\vec{S}$$

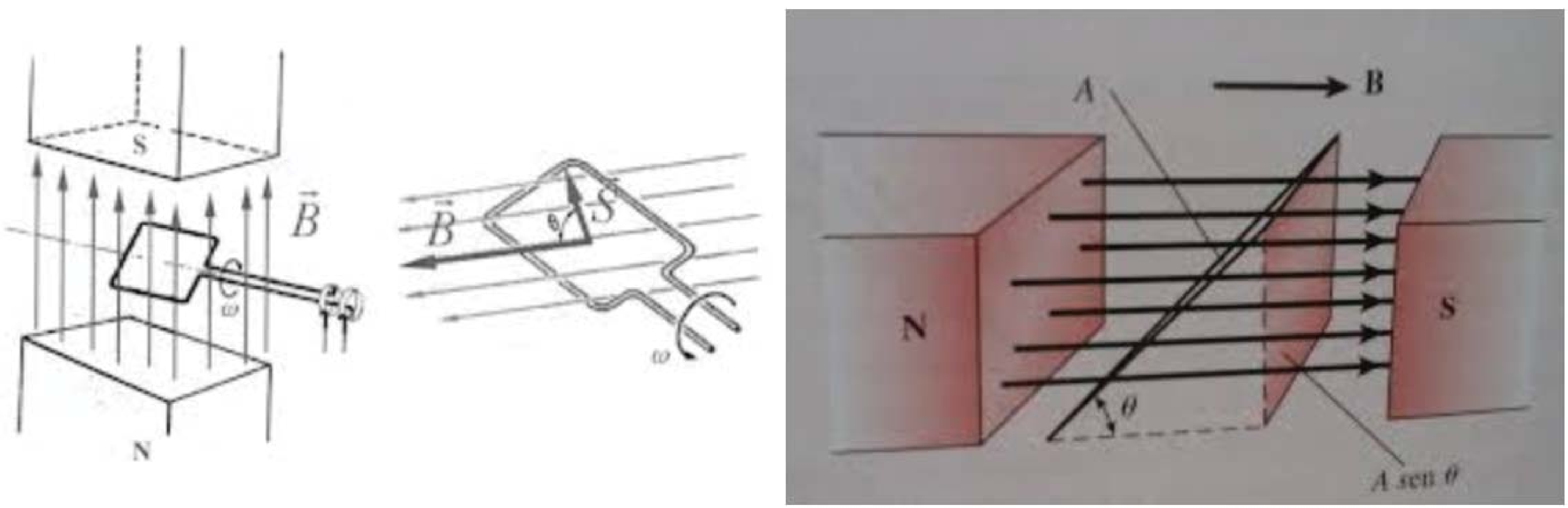


Figure 1 Magnetic flux

Magnetic Flux Density

- A magnetic flux line is a path to which $\vec{B}$ is tangential a every point on the line.

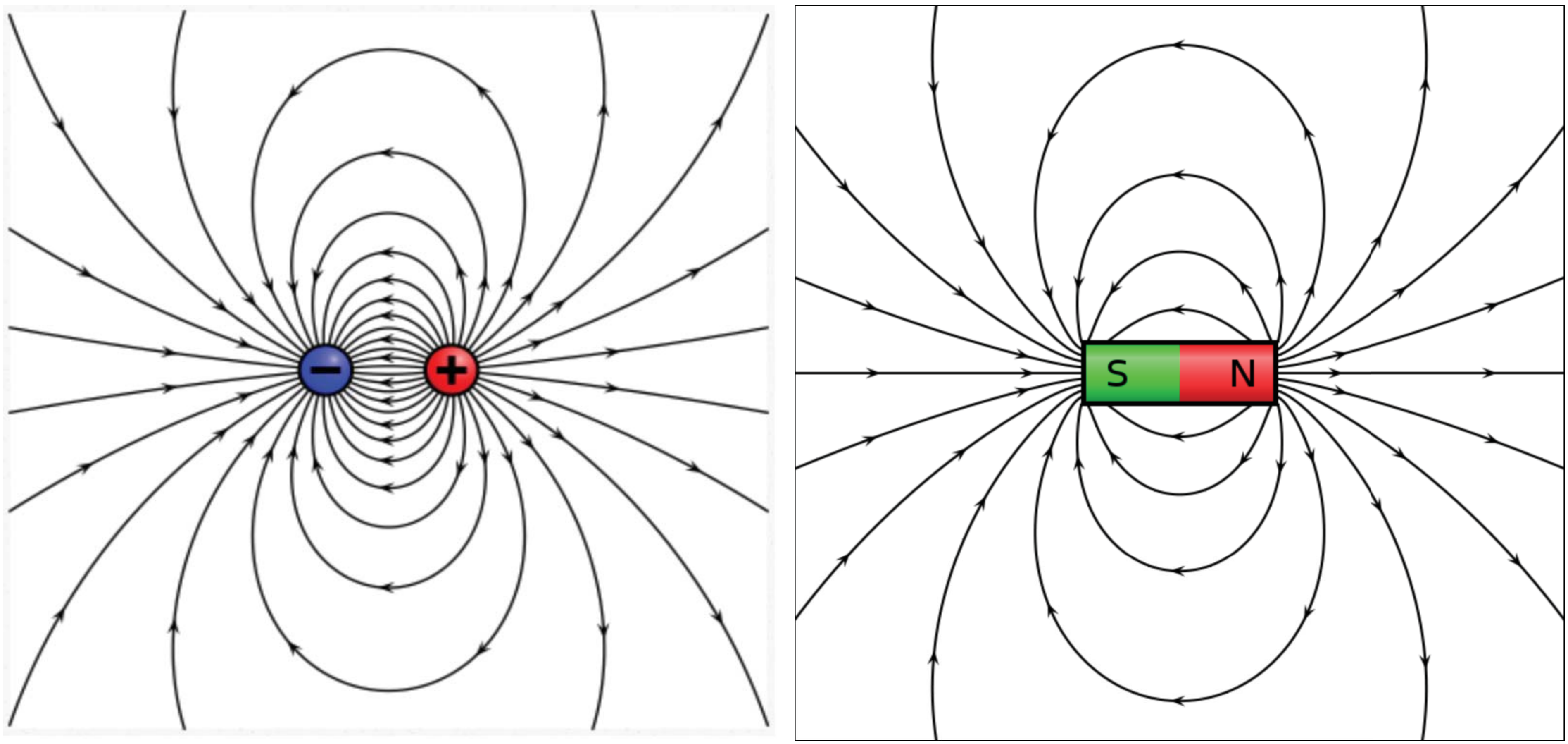


Figure 2: a) Electric Dipole and b) Magnetic Dipole

Magnetic Flux Density

- In an electrostatic field, the flux passing through a closed surface is te same as the charge enclosed; that is $\psi = \oint_S \vec{E} \cdot d\vec{S} = Q$.
- Thus it is possible to have an isolate electric charge as shown in Figure 3a).
- Which also reveals that electric flux lines are not necessarily closed.

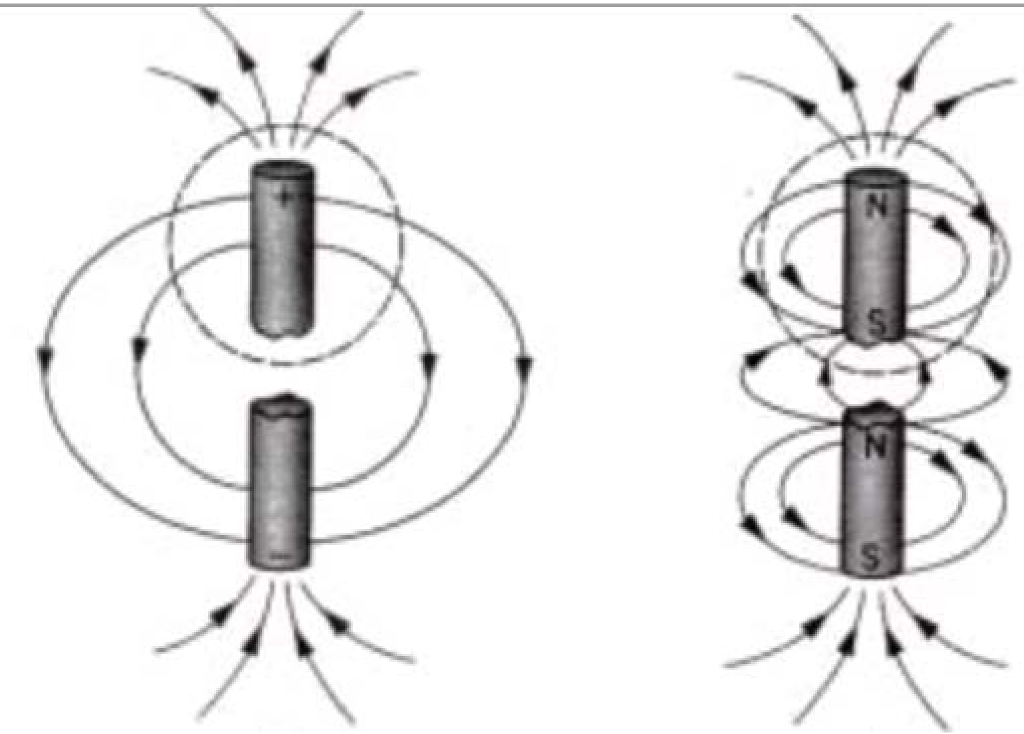


Figure 3: a) Isolate electric charge and b) Magnetic Dipole

Magnetic Flux Density

- Unlike electric flux flied, magnetic flux lines always close upon themselves as in Figure 3b).
- This is because it is not possible to have isolated magnetic poles (or magnetic charges).
- And isolated magnetic charge does not exist. Thus the total flux through a closed surface in a magnetic field must be zero; that is, $\psi = \oint_S \vec{B} \cdot d\vec{S} = 0$.

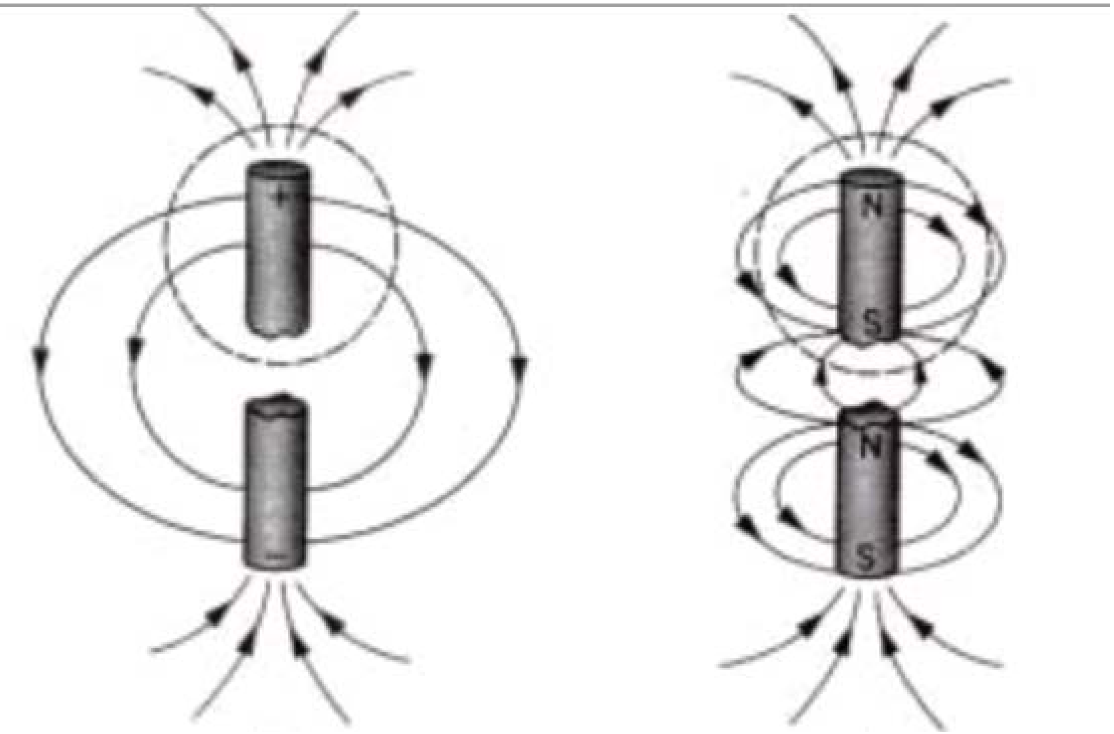


Figure 3: a) Isolate electric charge and b) Magnetic Dipole

Magnetic Flux Density

- $$\psi = \oint_S \vec{B} \cdot d\vec{S} = 0 \tag{4}$$

- This equation is referred to as the law of conservation of magnetic flux or Gauss's law for magnetostatic fields.

- By applying the divergence theorem, we obtain

  $$\psi = \oint_S \vec{B} \cdot d\vec{S} = \int_V \nabla \cdot \vec{B} dV = 0 \tag{5}$$

  or

  $$\nabla \cdot \vec{B} = 0 \tag{6}$$

MAXWELL'S EQUATIONS FOR STATICS FIELDS

Maxwell's equations for statics fields

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot \vec{D} = \rho_\nu$	$\oint_S \vec{D} \cdot d\vec{S} = \int_\nu \rho_\nu d\nu$	Gauss's law
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{S} = 0$	Nonexistence of magnetic monopole
$\nabla \times \vec{E} = 0$	$\oint_\ell \vec{E} \cdot d\vec{\ell} = 0$	Conservative nature of electrostatic field
$\nabla \times \vec{H} = \vec{J}$	$\oint_\ell \vec{H} \cdot d\vec{\ell} = \oint_S \vec{J} \cdot d\vec{S}$	Ampère's law

MAGNETIC SCALAR AND VECTOR POTENTIALS

Magnetic scalar and vector potentials

- We recall that some electrostatics field problems were simplified by relating the electric potential V to the electric field intensity $\vec{E}$ ($\vec{E} = -\nabla V$).

- Similarly, we can define a potential associated with magnetostatic field $\vec{B}$.

- In fact, the magnetic potential could be scalar V_m or vector $\vec{A}$.

- To define V_m and $\vec{A}$ involves recalling two important identities

$$\nabla \times (\nabla V) = 0 \tag{1}$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \tag{2}$$

which must always hold for any scalar field V and vector field $\vec{A}$.

Magnetic scalar and vector potentials

- Just as $\vec{E} = -\nabla V$, we define the *magnetic scalar potential* V_m (in amperes) as related to $\vec{H}$ according to

$$\vec{H} = -\nabla V_m \qquad \text{if} \qquad \vec{J} = 0 \tag{3}$$

- The condition attached to this equation is importan. Combining eq. (3) and $\nabla \times \vec{H} = \vec{J}$ give

$$\vec{J} = \nabla \times \vec{H} = \nabla \times (-\nabla V_m) = 0 \tag{4}$$

  since V_m must satisfy the condition in eq. (1).

- Thus the magnetic scalar potential V_m is only defined in a region where $\vec{J} = 0$ as eq. (3).

- We should also note that V_m satisfies Laplace's equation just V does for electrostatic field; hence,

$$\nabla^2 V_m = 0, \qquad (\vec{J} = 0) \tag{5}$$

Magnetic scalar and vector potentials

- We known that for magnetostatic field $\nabla \cdot \vec{B} = 0$.

- To satisfy last eq. and eq. (2) simultaneously, we can define the *vector magnetic potential* $\vec{A}$ (in Wb/m) such that

$$\vec{B} = \nabla \times \vec{A} \tag{6}$$

- Just as we define

$$V = \int \frac{dQ}{4\pi\varepsilon_0 r} \tag{7}$$

Magnetic scalar and vector potentials

- We can define

$$\vec{A} = \int_{\ell} \frac{\mu_o I \, d\vec{\ell}}{4\pi R} \quad \text{for line charge} \tag{8}$$

$$\vec{A} = \int_{S} \frac{\mu_o \vec{K} \, dS}{4\pi R} \quad \text{for surface current} \tag{9}$$

$$\vec{A} = \int_{\nu} \frac{\mu_o \vec{J} \, d\nu}{4\pi R} \quad \text{for volume current} \tag{10}$$

Magnetic scalar and vector potentials

- The magnetic flux through a surface S is given by

$$\psi = \int_S \vec{B} \cdot d\vec{S} \tag{11}$$

where the magnetic flux ψ is in Webers (Wb) and the magnetic flux density is in Webers per square meter (Wb/m^2) or Tesla (T).

- The *vector magnetic potential* $\vec{A}$ (in Wb/m) such that

$$\vec{B} = \nabla \times \vec{A} \tag{12}$$

- By substituting eq. (12) in to eq.(11) and applying Stokes's theorem, we obtain

$$\psi = \int_S \vec{B} \cdot d\vec{S} = \int_S (\nabla \times \vec{A}) \cdot d\vec{S} = \oint_\ell \vec{A} \cdot d\ell \tag{13}$$