Universidad Autónoma de Baja California
Facultad de Ingeniería, Arquitectura y Diseño

INGENIERÍA EN NANOTECNOLOGÍA

ETAPA DISCIPLINARIA

APUNTES

33543 CAMPOS ELECTROMAGNÉTICOS

Prof. E. Efrén García G.

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ETAPA DISCIPLINARIA Ingeniería en Nanotecnología

2 de febrero de 2021

33543 CAMPOS ELECTROMAGNÉTICOS

GRUPO: 741
Ciclo Escolar 2021-1

HORARIO DE CLASE:

Martes	*T*	*9:00 a 11:00 hrs*	---
Viernes	*C*	*11:00 a 12:00 hrs*	---
Viernes	*T*	*12:00 a 13:00 hrs*	---

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Propósito de la Unidad de Aprendizaje
La unidad de aprendizaje de Campos Electromagnéticos tiene como finalidad proporcionar al estudiante el marco teórico clásico del concepto de Onda Electromagnética (OE) como solución a la ecuación de onda para diferentes materiales; lo cual le permite utilizar los conceptos de la Teoría Electromagnética para su aplicación en el diseño de dispositivos nanoestructurados, cuyo funcionamiento se fundamente en efectos del electromagnetismo.

Competencia de la Unidad de Aprendizaje
Aplicar los conceptos de la Teoría Electromagnética, para dar solución a problemas de índole electromagnéticos que involucren el diseño de dispositivos nanoestructurados, a través de las técnicas y formalismo matemático basado en las ecuaciones de Maxwell, con actitud honesta, creativa y con buena disposición al trabajo colaborativo.

Contenido:

1.	**Campos Eléctricos.**	
	1.1	Fuerzas de Coulomb e Intensidad de Campo Eléctrico.
	1.2	Flujo Eléctrico y Ley de Gauss.
	1.3	Trabajo, Energía y Potencial.
	1.4	Corriente, Conductores y Capacitancia.
	1.5	Ecuaciones de Poisson y Laplace.

2.	**Campos Magnéticos.**	
	2.1	Ley de Biot-Savart.
	2.2	Ley de Ampère.
	2.3	Densidad de Flujo Magnético y ley de Gauss.
	2.4	Inductancia y Energía Magnética.

3.	**Ecuaciones de Maxwell.**	
	3.1	Ley de Faraday y fem's inducidas.
	3.2	Ley de Ampère y Corriente de Desplazamiento.
	3.3	Condiciones de Frontera.
	3.4	Ecuaciones de Maxwell.

4.	**Ondas Electromagnéticas.**	
	4.1	Ecuación de Onda.
	4.2	Propagación en diferentes medios.
	4.3	Condiciones de frontera para incidencia normal.
	4.4	Incidencia oblicua y Ley de Snell.

Evaluación:

Calificación mínima aprobatoria del curso	**60.0 (SESENTA)**

Criterios de evaluación:

1. Examen escrito individual: a. 1er Examen Parcial b. 2do Examen Parcial	 15% 15%
2. 100% de Asistencia y participación:	10%
3. Trabajo de investigación (1 – 2):	10%
4. Tareas semanales:	50%
TOTAL :	100%
5. El Examen Ordinario se exenta si el promedio general del curso corresponde a la calificación mínima aprobatoria (60.0). Si se presenta el examen ordinario, la calificación final corresponde a la calificación obtenida en dicho examen.	

6.	El alumno tendrá derecho a presentar el examen ordinario, si cubre el 80% o más de asistencias en clases impartidas.
7.	El alumno tendrá derecho a examen extraordinario, si cubre el 60% o más de asistencias en clases impartidas.

Bibliografía recomendada:

1.	Electromagnetics – Third Edition Joseph A. Edminister Mahmood Nahvi McGraw Hill – Schaum's ISBN 978-0-07-163235-5

2.	Elements of Electromagnetics Matthew N. O. Sadiku International Fourth Edition Oxford University Press ISBN 978-0-19-531519-6

3.	Electromagnetic Theory Julius Adams Stratton Massachusetts Institute of Technology ISBN 9781446517390

4.	The Electromagnetic Field Albert Shadowitz Dover Publications, Inc. New York ISBN 978-0-486-65660-1

5.	Electromagnetics Crash Course Joseph A. Edminister Schaum's easy outlines McGraw Hill ISBN 0-07-139879-1

6.	A Student's Guide to Maxwell's Equations Daniel Fleisch Cambridge University Press ISBN 978-0-521-70147-1

7.	Div, Grad, Curl, and all that H.M. Schey

	An informal Text on Vector Calculus W. W. Norton & Company New York - London

8.	http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html
9.	Maxwell's Equations of Electrodynamics An Explanation David W. Ball SPIE 2014

Profesor:

E. Efrén García G.

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Representante de Grupo:

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ELECTROSTATIC FIELDS

Introduction

Before we commence our study of electrostatics, it might be helpful to examine briefly the importance of such study. Electrostatics is fascinating subject that has grown up in diverse areas of application.

- Electric power transmission, X-ray machine, and lightning protection are associate with strong electric field and will required a knowledge of electrostatics to understand and design suitable equipment.

- The device used in solid-sate electronic are based on electrostatic. These include resistors, capacitors, and device such as bipolar and field effect transistors, which are based on control of electron motion by electrostatic fields.

- Almost all peripheral devices, with de exception of magnetic memory, are based on electrostatic fields. Toch pad, capacitance keyboards, chatode-ray tube, liquid crystal display, and electrostatics printers are typical examples.

- In medical work, diagnosis is often carried out with the aid of electrostatic, as incorporated in electrocardiograms, electroencephalograms, and other recording of the electrical activity of organs include eyes, ears, and stomach.

- In industry, electrostatics is applied in a variety of forms such as paint spraying, electrodeposition, electrochemical machining, and separation of fine particles.

- Electrostatics is used in agriculture to sort seed, for direct spraying of plant, to mesure te moisture content in crops, to spin cotton, and for speed baking bread and smoking meat.

Coulomb's law and Field Intensity

- Coulomb's law is and experimental law formulated in 1785 by Charles Augustin de Coulomb, then colonel in the France army.
- One coulomb is approximately equivalent to 6×10^{18} electron; it is very large unit of charge because one electron charge $e = -1.6019 \times 10^{-19}$ C.
- Coulomb's law states that the Force F between two carge Q_1 and Q_2 is:
    1. Along the line joining them
    2. Directly proportional to the product $Q_1 Q_2$ of the charges
    3. Inversely proportional to the square of distance R between them.

Expressed mathematically,

$$F = \frac{k Q_1 Q_2}{R^2} \tag{1}$$

where k is the proportionality constant whose value depend on the choice of system of units. In **SI** unit, charges Q_1 and Q_2 are in coulombs (C), the distance R is in meters (m), and the force F is in newtons (N) so that $k = 1/4\pi\varepsilon_0$. The constant ε_0 as the *permittivity of free space* (in farads per meter) and has the value
$\varepsilon_0 = 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi}$ F/m or $k = \frac{1}{4\pi\varepsilon_0}$ m/F

Coulomb's law and Field Intensity

$$F = \frac{kQ_1Q_2}{R^2} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2} \tag{2}$$

If point charges Q_1 and Q_2 are located at point having position vectors $\vec{r}_1$ and $\vec{r}_2$, then the force $\vec{F}_{12}$ on Q_2 due Q_1, is given by

$$\vec{F}_{12} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\hat{a}_{R_{12}} \tag{3}$$

where

$$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1 \tag{4}$$

$$R = \mid \vec{R}_{12} \mid \tag{5}$$

$$\hat{a}_{R_{12}} = \frac{\vec{R}_{12}}{R} \tag{6}$$

Then, we write eq.(3) as

$$\vec{F}_{12} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^3}\vec{R}_{12} \tag{7}$$

or

$$\vec{F}_{12} = \frac{Q_1Q_2(\vec{r}_2 - \vec{r}_1)}{4\pi\varepsilon_0 \mid \vec{r}_2 - \vec{r}_1 \mid^3} \tag{8}$$

Coulomb's law and Field Intensity

It is worthwhile to note that

- As shown, the force $\vec{F}_{21} = |\vec{F}_{12}|\, \widehat{a}_{R_{21}} = |\vec{F}_{12}|\, (-\widehat{a}_{R_{12}})$

  or $\vec{F}_{12} = -\vec{F}_{21}$

  since $\widehat{a}_{R_{21}} = -\widehat{a}_{R_{12}}$

- Like charges (charges of the same sign) repel each other, while unlike charge attract.

- The distance R between the charged bodies Q_1 and Q_2 must be large compared with the linear dimensions of the bodies; that is, Q_1 and Q_2 must be point charge.

- Q_1 and Q_2 must be static (at rest).

- The sign of Q_1 and Q_2 must be taken into account in eq.(3) for likes charges $Q_1 Q_2 > 0$. For unlike charges $Q_1 Q_2 < 0$.

Coulomb's law and Field Intensity

If we have more than two point charges, we can use the *principle of superposition* to determine the force on a particular charge. The principle state that if there are N charges $Q_1, Q_2, ..., Q_N$ located, respectively, at points with position vectors $\vec{r}_1, \vec{r}_2, ..., \vec{r}_N$ the resultant force $\vec{F}$ on a charge Q located at point $\vec{r}$ is the vector sum of the forces exerted on Q by each of the charge $Q_1, Q_2, ..., Q_N$. Hence:

$$\vec{F} = \frac{QQ_1(\vec{r} - \vec{r}_1)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_1 \mid^3} + \frac{QQ_2(\vec{r} - \vec{r}_2)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_2 \mid^3} + ... + \frac{QQ_N(\vec{r} - \vec{r}_N)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_N \mid^3} \tag{9}$$

or

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{Q_k(\vec{r} - \vec{r}_k)}{\mid \vec{r} - \vec{r}_k \mid^3} \tag{10}$$

Coulomb's law and Field Intensity

We can now introduce the concept of *electric field intensity*:

The **electric field intensity** (or **electric field strength**) $\vec{E}$ is the force per unit charge when placed in an electric field.
Thus

$$\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q} \tag{11}$$

or simply

$$\vec{E} = \frac{\vec{F}}{Q} \tag{12}$$

For $Q > 0$, the electric field intensity $\vec{E}$ is obviously in the direction of the force $\vec{F}$ and is measured in newtons per coulomb or volts per meter. The electric field intensity at point $\vec{r}$ due to a point charge located at $\vec{r\prime}$ is readily obtained from eqs.(3) and (12) as

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\widehat{a}_R = \frac{Q(\vec{r} - \vec{r\prime})}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r\prime} \mid^3} \tag{13}$$

Coulomb's law and Field Intensity

For N point charges $Q_1, Q_2, ..., Q_N$ located at $\vec{r}_1, \vec{r}_2, ..., \vec{r}_N$, the electric field intensity at point $\vec{r}$ is obtained from eqs. (10) and (12) as

$$\vec{E} = \frac{Q_1(\vec{r} - \vec{r}_1)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_1 \mid^3} + \frac{Q_2(\vec{r} - \vec{r}_2)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_2 \mid^3} + ... + \frac{Q_N(\vec{r} - \vec{r}_N)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_N \mid^3} \tag{14}$$

or

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{Q_k(\vec{r} - \vec{r}_k)}{\mid \vec{r} - \vec{r}_k \mid^3} \tag{15}$$

Electric Field due to continuous charge distribution

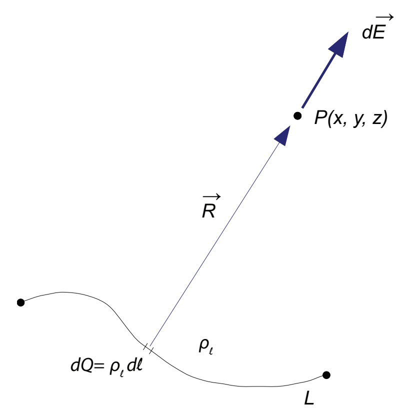
So far we have considered only forces and electric field due to point charges, which are essentially charges occupying very small physical space. It is also possible to have continuous charge distribution along line, surface, or in a volume.

It is customary to denote the line charge density, surface charge density, and volume charge density by ρ_ℓ (in C/m), ρ_s (in C/m^2), and ρ_v (in C/m^3). These must not be confused with ρ (without subscript), used for radial distance in cylindrical coordinates.

Electric Field due to continuous charge distribution

- For a linear charge density ρ_ℓ (C/m), the elemental charge $dQ = \rho_\ell \, d\ell$ and the differential field at a point P would be

$$d\vec{E} = \frac{\rho_\ell d\ell}{4\pi\varepsilon_0 R^2}\,\widehat{a}_R \tag{16}$$



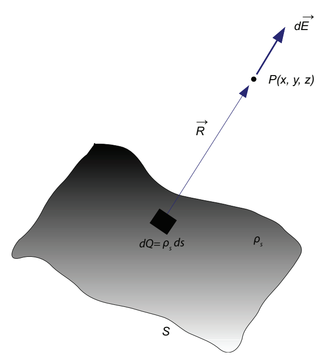
The total field at the observation point P is obtained by integrating over the line or curve L

$$\vec{E} = \int_L \frac{\rho_\ell \,\widehat{a}_R}{4\pi\varepsilon_0 R^2} d\ell \tag{17}$$

Electric Field due to continuous charge distribution

- For a surface charge density ρ_s (C/m^2), the elemental charge $dQ = \rho_s\,dS$ and the differential field at a point P would be

$$d\vec{E} = \frac{\rho_s ds}{4\pi\varepsilon_0 R^2}\,\widehat{a}_R \tag{18}$$



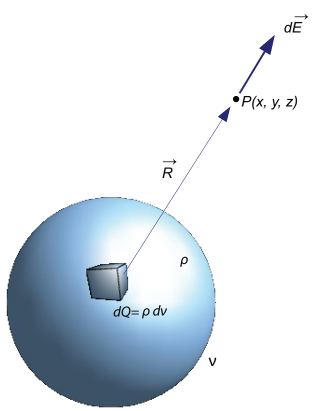
The total field at the observation point P is obtained by integrating over surface S

$$\vec{E} = \int_s \frac{\rho_s\,\widehat{a}_R}{4\pi\varepsilon_0 R^2} dS \tag{19}$$

Electric Field due to continuous charge distribution

- For a volume charge density ρ (C/m^3), the elemental charge $dQ = \rho_v \, dv$ and the differential field at the point P would be

$$d\vec{E} = \frac{\rho_v dv}{4\pi\varepsilon_0 R^2}\,\widehat{a}_R \tag{20}$$



The total field at the observation point P is obtained by integrating over volume v

$$\vec{E} = \int_v \frac{\rho_v \, \widehat{a}_R}{4\pi\varepsilon_0 R^2} dv \tag{21}$$

Problems

1. Point charges 1 mC and -2 mC are located at $(3, 2, -1)$ and $(-1, -1, 4)$ respectively. Calculate the electric force on a 10 nC charge located at $(0, 3, 1)$ and the electric field intensity at that point.

2. Point charges 5 nC and -2 nC are located $(2, 0, 4)$ and $(-3, 0, 5)$, respectively.
   (a) Determine the force on a 1 nC point charge located at $(1, -3, 7)$.
   (b) Find the electric field $\vec{E}$ at $(1, -3, 7)$.

3. A circular ring of radius a carries a uniform charge ρ_ℓ C/m and is placed on the xy-plane with axis the same as the z-axis.
   (a) Show that
   $$\vec{E}(0, 0, h) = \frac{\rho_\ell a h}{2\epsilon_0 [h^2 + a^2]^{3/2}} \, \widehat{a}_z$$
   (b) What values of h gives the maximum value of $\vec{E}$?
   (c) If the total charge on the ring is Q, find $\vec{E}$ as $a \to 0$.

4. The finite sheet $0 \leq x \leq 1$, $0 \leq y \leq 1$ on the $z = 0$ plane has a charge density $\rho_s = xy(x^2 + y^2 + 25)^{3/2}$ nC/m^2. Find
   (a) The total charge on the sheet.
   (b) The electric field at $(0, 0, 5)$.
   (c) The force experimented by a -1 mC charge located at $(0, 0, 5)$.

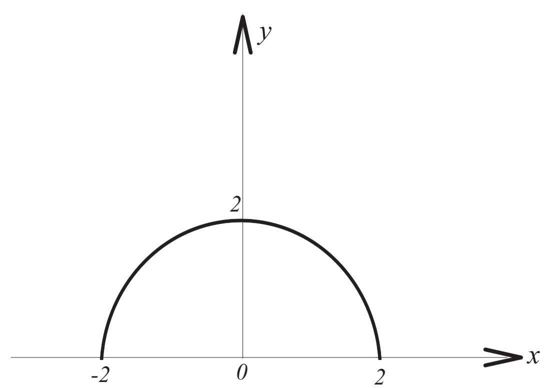
5. A square plate described by $-2 \leq x \leq 2$, $-2 \leq y \leq 2$, $z = 0$ carries a charge 12 $|\, y \,|$ mC/m^2. Find the total charge on the plate and the electric field intensity at $(0, 0, 10)$.

6. Planes $x = 2$ and $y = 3$, respectively, carry charges 10 nC/m^2 and 15 nC/m^2. If the line $x = 0$, $z = 2$ carries charge 10π nC/m, calculate $\vec{E}$ at $(1, 1, -1)$ due to the three charge distributions.

7. Planes $x = 2$ and $y = 3$, respectively, carry charges 10 nC/m^2 and 15 nC/m^2. If the line $x = 0$, $z = 2$ carries charge 10π nC/m is rotated through 90^0 about the point $(0, 2, 2)$ so that it become $x = 0$, $y = 2$, find $\vec{E}$ at $(1, 1, -1)$.

8. Charges $+Q$ and $+3Q$ separated by a distance 2 m. A third charge is located such that the electrostatic system is in equilibrium. Find the location and the value of the third charge in terms of Q.

9. A point charge Q is located at point $P(0, -4, 0)$, while a 10 nC charge is uniformly distributed along a circular ring as shown in the next Figure. Find the value of Q such that $E(0, 0, 0) = 0$.



10. (a) Show that the electric field at point $(0, 0, h)$ due to the rectangle described by $-a \leq x \leq a$, $-b \leq y \leq b$, $z = 0$ carrying uniform charge ρ_s C/m^2 is

$$\vec{E}(0, 0, h) = \frac{\rho_s}{\pi \epsilon_0} tan^{-1}[\frac{ab}{h(a^2 + b^2 + h^2)^{1/2}}]\, \hat{a}_z$$

(b) If $a = 2$, $b = 5$, $\rho_s = 10^{-5}$, find the total charge on the plate and the electric field intensity at $(0, 0, 10)$.

ELECTRIC FLUX DENSITY

Electric Field Intensity

The electric field intensity at point $\vec{r}$ due to a point charge located at $\vec{r'}$ is readily obtained from eq.(1),

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R = \frac{Q(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r'} \mid^3} \tag{1}$$

For N point charges $Q_1, Q_2, ..., Q_N$ located at $\vec{r}_1, \vec{r}_2, ..., \vec{r}_N$, the electric field intensity at point $\vec{r}$ is obtained from eqs. (2) and (3) as

$$\vec{E} = \frac{Q_1(\vec{r} - \vec{r}_1)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_1 \mid^3} + \frac{Q_2(\vec{r} - \vec{r}_2)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_2 \mid^3} + ... + \frac{Q_N(\vec{r} - \vec{r}_N)}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_N \mid^3} \tag{2}$$

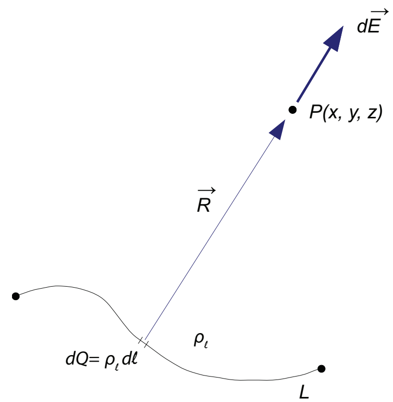
or

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k(\vec{r} - \vec{r}_k)}{\mid \vec{r} - \vec{r}_k \mid^3} \tag{3}$$

Electric Field Intensity

- For a linear charge density ρ_ℓ (C/m), the elemental charge $dQ = \rho_\ell \, d\ell$ and the differential field at a point P would be

$$d\vec{E} = \frac{\rho_\ell d\ell}{4\pi\varepsilon_0 R^2}\,\widehat{a}_R \tag{4}$$



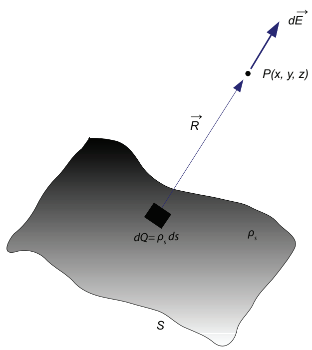
The total field at the observation point P is obtained by integrating over the line or curve L

$$\vec{E} = \int_L \frac{\rho_\ell \,\widehat{a}_R}{4\pi\varepsilon_0 R^2} d\ell \tag{5}$$

Electric Field Intensity

- For a surface charge density ρ_s (C/m^2), the elemental charge $dQ = \rho_s \, dS$ and the differential field at a point P would be

$$d\vec{E} = \frac{\rho_s ds}{4\pi\varepsilon_0 R^2}\,\widehat{a}_R \tag{6}$$



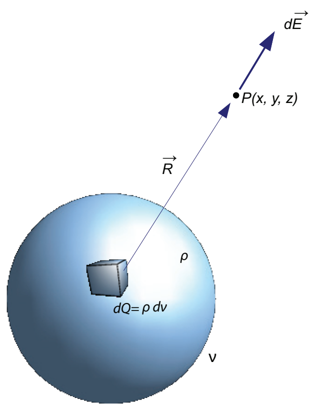
The total field at the observation point P is obtained by integrating over surface S

$$\vec{E} = \int_s \frac{\rho_s\,\widehat{a}_R}{4\pi\varepsilon_0 R^2} dS \tag{7}$$

Electric Field Intensity

- For a volume charge density ρ (C/m^3), the elemental charge $dQ = \rho_v \, dv$ and the differential field at the point P would be

$$d\vec{E} = \frac{\rho_V dV}{4\pi\varepsilon_0 R^2}\,\widehat{a}_R \tag{8}$$



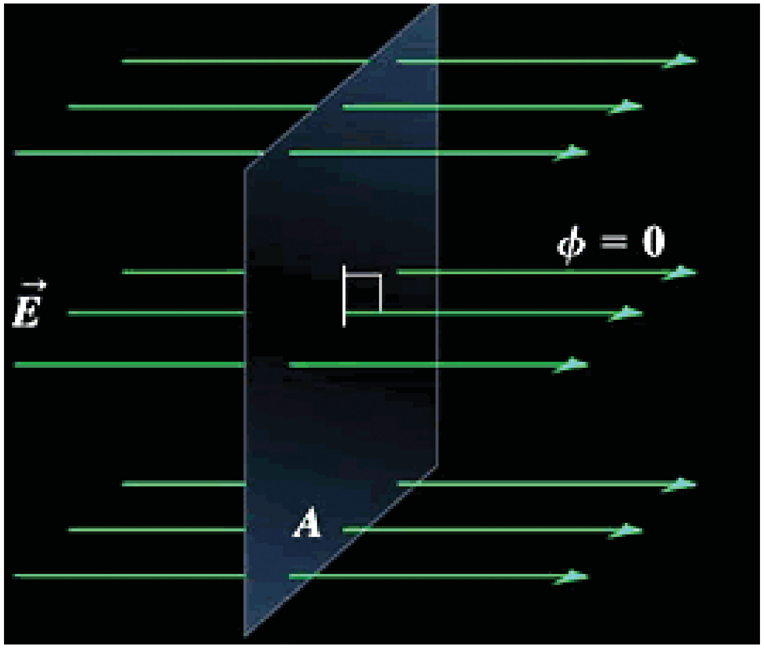
The total field at the observation point P is obtained by integrating over volume v

$$\vec{E} = \int_V \frac{\rho_V \,\widehat{a}_R}{4\pi\varepsilon_0 R^2} dV \tag{9}$$

Electric Flux Density

- Eqs. (1) to (9) show that the electric field intensity is dependent on the medium in with the charge is placed (free space in this case ε_0).

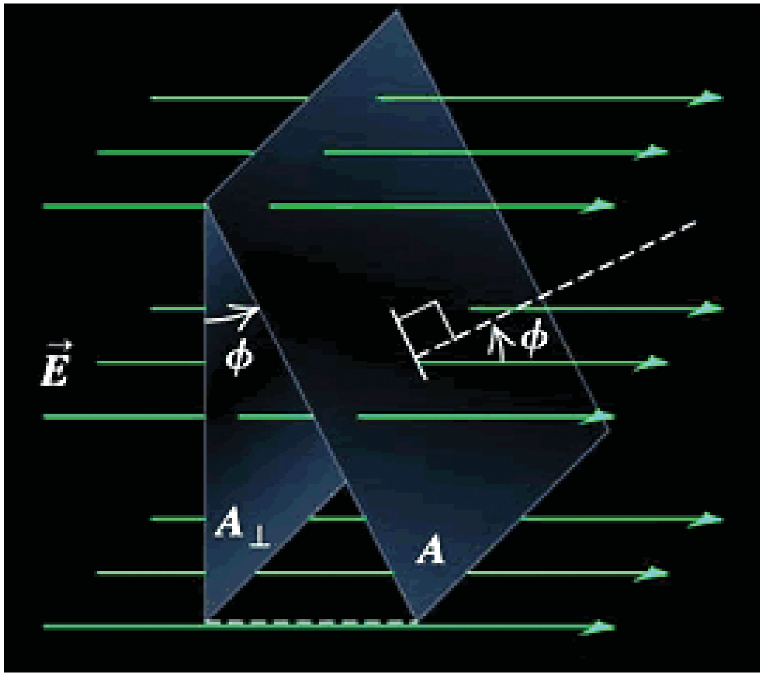
- Electric field $\vec{E}$ is perpendicular to area A; the angle between $\vec{E}$ and line perpendicular to the surface is zero.



- The flux is $\Phi_E = EA$.

Electric Flux Density

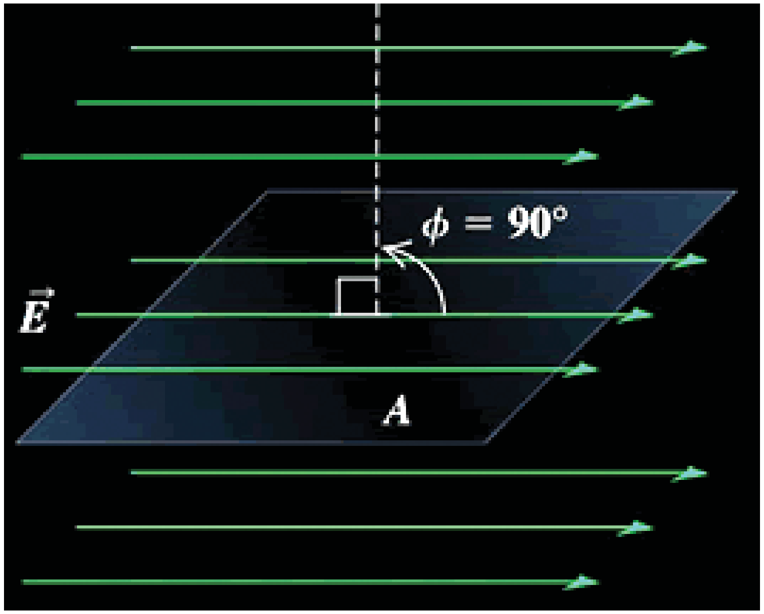
- Area A is tilted at an angle Φ from the perpendicular to $\vec{E}$.



- The flux is $\Phi_E = EA\cos(\Phi)$.

Electric Flux Density

- Area A is parallel to $\vec{E}$ (tilted at 90^0 from the perpendicular to $\vec{E}$).



- The flux is $\Phi_E = EA\cos(90^0) = 0$.

Electric Flux Density

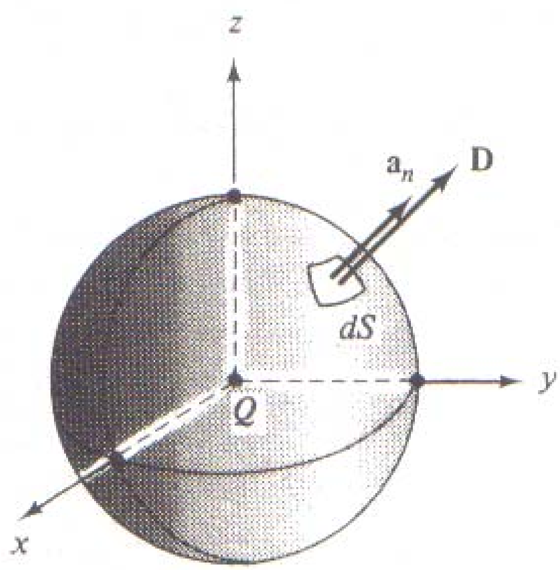
- Suppose a new vector field $\vec{D}$ is defined by

$$\vec{D} = \varepsilon_0 \vec{E} \tag{10}$$

- The electric flux Ψ in terms of $\vec{D}$ is

$$\Psi = \int_s \vec{D} \cdot d\vec{S} \tag{11}$$

- The electric flux is measured in coulombs. The vector field $\vec{D}$ is called the electric flux density and is measured in coulombs per square meter.



GAUSS'S LAW - MAXWELL'S EQUATION

Gauss's Law

- Gauss's law constitutes one of the fundamental laws of electromagnetism. **Gauss's law** state that the total electric flux Ψ through any *closed* surface is equal to the total charge enclosed by that surface

- Thus

$$\Psi = Q_{enc} \tag{12}$$

- That is,

$$\Psi = \oint_s d\Psi = \oint_s \vec{D} \cdot d\vec{S} =$$

$$\textit{total charge enclosed } Q = \int_V \rho_V dV \tag{13}$$

- or

$$Q = \oint_s \vec{D} \cdot d\vec{S} = \int_V \rho_V dV \tag{14}$$

Gauss's Law

- By applying divergence theorem to the middle term in eq (14), we have

$$\oint_s \vec{D} \cdot d\vec{S} = \int_V \nabla \cdot \vec{D} dV \tag{15}$$

- Comparing the two integrals results is

$$\rho_V = \nabla \cdot \vec{D} \tag{16}$$

- Eq (16) is the first of the four *Maxwell's* equations to be derived.
- This equation states that the volume charge density is the same as the divergence of the electric flux density.
- Equations (14) and (16) are basically stating *Gauss's law* in different ways; eq. (14) is the integral form, whereas eq. (16) is the differential or point form of Gauss's law.
- Gauss's law provides an easy means of finding $\vec{E}$ or $\vec{D}$ for symmetrical charge distributions such as a point charge, an infinite line charge, an infinite cylindrical surface charge, and a spherical distribution of charge.

Gauss's Law

- The *divergence of a static field* is used to determine when a region has *sources* (*net positive charge*) or *sinks* (*net negative charge*). By definition, the divergence of the electric lux density at a point P is

$$\text{div}\vec{D} = \nabla \cdot \vec{D} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{D} \cdot d\vec{S}}{\Delta V} = \lim_{\Delta V \to 0} \frac{Q_{enclosed}}{\Delta V} = \rho \tag{17}$$

where S is the boundary of ΔV.

- For a general vector $\vec{A}$, the definitions for the divergence in the three coordinate systems of interes are:

$$\text{Cartesian}: \qquad \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{18}$$

$$\text{Cylindrical}: \qquad \nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{19}$$

$$\text{Spherical}: \qquad \nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial \phi} \tag{20}$$

APPLICATIONS OF GAUSS'S LAW

Gauss's Law

- For applying Gauss's law to calculate the electric field involves first knowing whether symmetry exist.
- Once it has been found that symmetric charge distribution exit, we construct a mathematical closed surface (known as Gaussian surface).
- The surface is chosen such that $\vec{D}$ is normal or tangential to the Gaussian surface.
- When $\vec{D}$ is normal to the surface, $\vec{D} \cdot d\vec{S} = DdS$ because $\vec{D}$ is constant on the surface.
- When $\vec{D}$ is tangential to the surface, $\vec{D} \cdot d\vec{S} = 0$.
- Thus we must choose a surface that has some of the symmetry exhibited by the charge distribution.

Problems

NOTE: The integral and point forms of Gauss' law are related by the *divergence theorem* given by

$$\Psi = \oint_S \vec{D} \cdot d\vec{S} = \int_V (\nabla \cdot \vec{D}) dV = \int_V \rho_V dV = Q_{\text{enclosed}} \tag{21}$$

where S is the closed surface boundary of the volume V.

1. **Example:** In the region $0 < r < 1$ m, $\vec{D} = (-2\text{x}10^{-4}/r)\,\hat{a}_r$ (C/m^2) and for $r > 1$ m, $\vec{D} = (-4\text{x}10^{-4}/r^2)\,\hat{a}_r$ (C/m^2) in spherical coordinates. Find the charge density in both region.

2. Charge in the form of a plane sheet with density $\rho_s = 40$ (μC/m^2) is located at $z = -0.5$ (m). A uniform line charge of $\rho_\ell = -6$ (μC/m) lies along the $y-$axis. What net flux crosses the surface of a cube 2 (m) on an edge, centered at the origin of coordinate system?

3. Determine the flux crossing a 1 mm by 1 mm area on surface of a cylindrical shell at $r = 10$ m, $z = 2$ m, $\psi = 53.3^o$ if $\vec{D} = 2x\,\hat{a}_x + 2(1-y)\,\hat{a}_y + 4z\,\hat{a}_z$.

ELECTRIC POTENTIAL

Electric Potential

- We can obtain the electric file intensity $\vec{E}$ due to a charge distribution from Coulomb's law in general or, when the charge distribution is symmetric, from Gauss's law.
- Another way of obtaining $\vec{E}$ is from the electric scalar potential V.
- In a sense, this way of finding $\vec{E}$ is easier because it is easier to handle scalars than vectors.
- Suppose we wish to move a point charge Q from point A to point B in a electric field $\vec{E}$ as show in Figure 1.

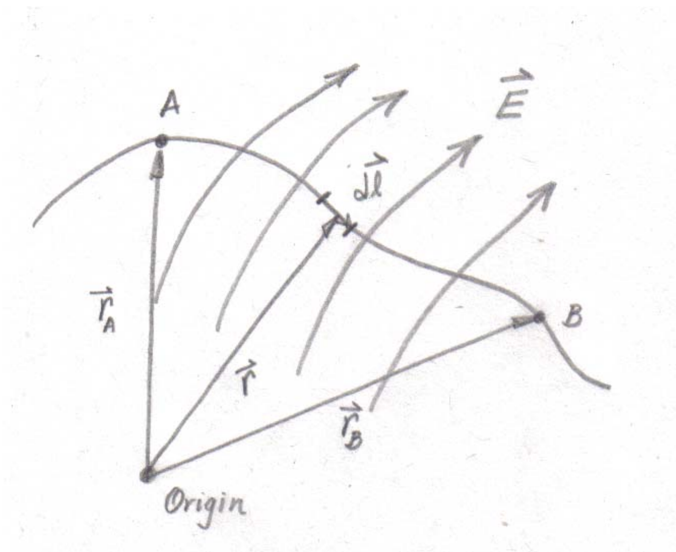


Figure 1 Displacement of point charge Q in an electric field $\vec{E}$.

Electric Potential

- From Coulomb's law, the force on Q is $\vec{F} = Q\vec{E}$ so that the *work done* in displacing the charge by $d\vec{\ell}$ is

$$dW = -\vec{F} \cdot d\vec{\ell} = -Q\vec{E} \cdot d\vec{\ell} \tag{1}$$

  The negative sing indicate that the work is being done by external agent.

- Thus the total work done, or the potential energy required, in moving Q from A to B, is

$$W = -Q \int_A^B \vec{E} \cdot d\vec{\ell} \tag{2}$$

- Dividing W by Q in eq. (2) gives the potential energy per unit charge. This quantity, denoted by V_{AB}, is known as the *potential difference* between points A and B. Thus

$$V_{AB} = \frac{W}{Q} = -\int_A^B \vec{E} \cdot d\vec{\ell} \tag{3}$$

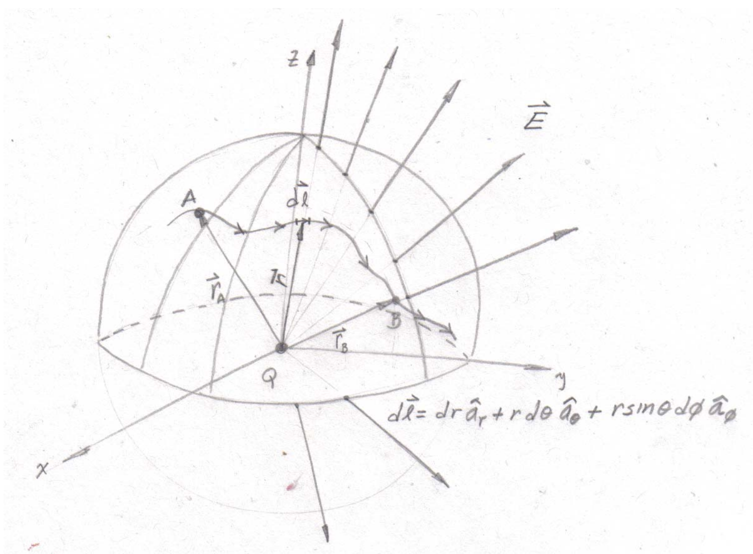
Electric Potential

Note that:

- In determining V_{AB}, A is the initial point while B is the final point.
- If V_{AB} is negative, there is a loss in potential energy in moving Q from A to B; this implies that the work is being done by the field.
- If V_{AB} is positive, there is a gain in potential energy in the movement; an external agent performs the work.
- V_{AB} is independent of the path taken (to be shown a little later).
- V_{AB} is measured in joules per meter, commonly referred to a volts (V).

Electric Potential

As an example, if the $\vec{E}$ field in Figure 1 is due to a point charge Q located at the origin, then



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \tag{4}$$

so eq.(3)

$$V_{AB} = -\int_A^B \vec{E}\cdot d\vec{\ell} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \cdot dr\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}[\frac{1}{r_B} - \frac{1}{r_A}] \tag{5}$$

Electric Potential

or

$$V_{AB} = V_B - V_A \tag{6}$$

where V_B and V_A are the *potentials* (or *absolute potentials*) at B and A, respectively.

- Thus the potential difference V_{AB} may be regarded as the potential at B with reference to A.

- Note from eq. (5) that because $\vec{E}$ points in the radial direction, any contribution from displacement in the θ or ϕ direction is wiped out by the dot product $\vec{E} \cdot d\vec{\ell} = E \cos\alpha d\ell = E dr$, where α is the angle between $\vec{E}$ and $d\vec{\ell}$. Hence the potential difference V_{AB} is independent of the path as asserted earlier.

- In general, vectors whose line integral does not depend on the path of integration are called conservative. This $\vec{E}$ is conservative.

Electric Potential

- The potential at any point ($r_B \to r$) due to charge Q located at the origin is

$$V = \frac{Q}{4\pi\varepsilon_0 r} \tag{7}$$

- The potential at any point is the potential difference between that point and a chosen point (or reference point) at which the potential is zero.

- In other words, by assuming zero potential at infinity, the potential at a distance r from the point charge is the work done per unit cherge by an external agent in transferring a test charge from infinity to that point. Thus

$$V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{\ell} \tag{8}$$

Electric Potential

- If the point charge Q in eq. (7) is not located at the origin but at a point whose position vector is $\vec{r'}$, the potential $V(x, y, z)$ or simply $V(\vec{r})$ at $\vec{r}$ becomes

$$V = \frac{Q}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r'} \mid} \tag{9}$$

- The superposition principle, wich we applied to electric field, applies to potential. For n point charges $Q_1, Q_2, ..., Q_n$ located at points with position vectors $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$, the potential at $\vec{r}$ is

$$V(\vec{r}) = \frac{Q_1}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_1 \mid} + \frac{Q_2}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_2 \mid} + ... + \frac{Q_n}{4\pi\varepsilon_0 \mid \vec{r} - \vec{r}_n \mid} \tag{10}$$

or

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{Q_k}{\mid \vec{r} - \vec{r}_k \mid} \quad \text{point charges} \tag{11}$$

Electric Potential

- For continuous charge distribution, we replace Q in eq. (11) with charge element $\rho_\ell \, d\ell$, $\rho_S \, dS$, or $\rho_V \, dV$ and the summation becomes an integration, so the potential at $\vec{r}$ become

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\rho_\ell(\vec{r'}) \, d\ell'}{|\, \vec{r} - \vec{r'} \,|} \quad \text{line charge} \tag{12}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_S(\vec{r'}) \, dS'}{|\, \vec{r} - \vec{r'} \,|} \quad \text{surface charge} \tag{13}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_V(\vec{r'}) \, dV'}{|\, \vec{r} - \vec{r'} \,|} \quad \text{volume charge} \tag{14}$$

where the primed coordinates are used customarily to denote source point location and the unprimed coordinates refer to field point (the point at which V is to be determined).

Electric Potential

The following points should be noted:

- We recall that in obtaining eqs. (7) to (14), the zero potential (reference) point has been chosen arbitrarily to be at infinity. If any other point is chosen as reference, eq.(9), for example, becomes

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C \tag{15}$$

  where C is a constant that is determined at the chosen point of reference. The same idea applies to (9) to (14).

- The potential at a point can be determined in two ways depending on whether the charge distribution or $\vec{E}$ is known. If the charge distribution is known, we use on of the (7) to (14) depending on the charge distribution.

- If $\vec{E}$ is known, we simply use

$$V = -\int \vec{E} \cdot d\vec{\ell} + C \tag{16}$$

- The potential difference V_{AB} can be found generally from

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{\ell} = \frac{W}{Q} \tag{17}$$

RELATIONSHIP BETWEEN $\vec{E}$ and V

Relationship between $\vec{E}$ and V

- The potential difference between points A and B is independent of the path taken. Hence

$$V_{BA} = -V_{AB}$$

that is, $V_{BA} + V_{AB} = \oint \vec{E} \cdot d\vec{\ell} = 0$ or

$$\oint_L \vec{E} \cdot d\vec{\ell} = 0 \tag{18}$$

This shows that the line integral of $\vec{E}$ along a closed path as shown in Figure 3, must be zero.

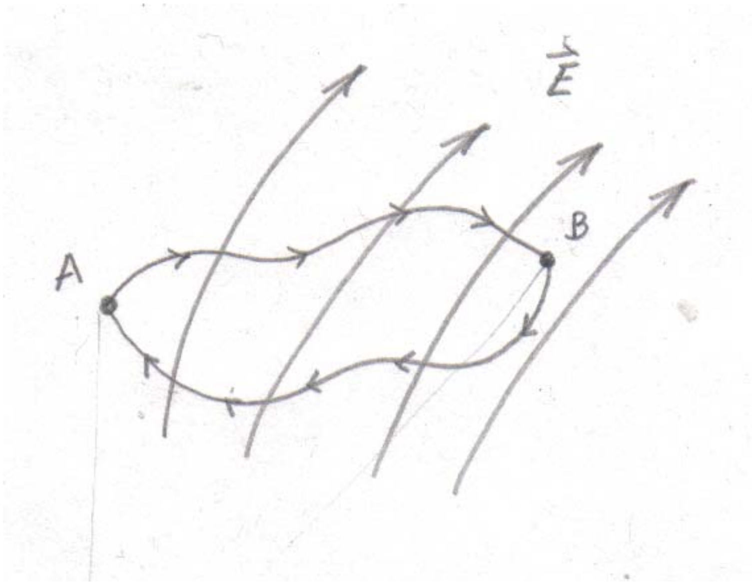


Figure 3 The conservative nature of an electric field.

Relationship between $\vec{E}$ and V

- Physically, this implies that no net work is done in moving a charge along a closed path in an electrostatic field.

- Applying Stokes's theorem to eq.(18) gives

$$\oint_L \vec{E} \cdot d\vec{\ell} = \int_s (\nabla \times \vec{E}) \cdot d\vec{S} = 0 \tag{19}$$

  or

$$\nabla \times \vec{E} = 0 \tag{20}$$

- Any vector field that satisfies eq.(19) or eq.(20) is said to be conservative, or irrotational.

- In other words, vectors whose line integral does not depend on the path of integration are called conservative vectors.

- Thus an electrostatic field is a conservative field. Equations (19) or (20) is referred as *Maxwell's equation* (the second Maxwell equation to be derived) for static electric field.

Relationship between $\vec{E}$ and V

- Equation (19) is the integral form, and eq. (20) is the differential form; they both depict the conservative nature of and electrostatic field.

- From the way we defined potential, $V = -\int \vec{E} \cdot d\vec{\ell}$, it follow that

$$dV = -\vec{E} \cdot d\vec{\ell} = -E_x dx - E_y dy - E_z dz \tag{21}$$

- But from calculus of multivariables, a total charge in $V(x, y, z)$ is the sum of partial charges with respect to x, y, z variables:

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \tag{22}$$

- Comparing eqs. (21) and (22)

$$\vec{E} = -\nabla V \tag{23}$$

that is, the electric field is the gradient of V.

Relationship between $\vec{E}$ and V

- The negative sign shows that the direction of $\vec{E}$ is opposite to the direction in which V increases; $\vec{E}$ is directed from higher to lower levels of V.
- Equation (23) shows another way to obtain $\vec{E}$ field apart from using Coulomb's or Gauss's law.
- One may wonder how one function V can possibly contain all the information that the three components of $\vec{E}$ carry.

Problems

1. An electrostatic field is given by $\vec{E} = \frac{x}{2} + 2y\,\widehat{a}_x + 2x\,\widehat{a}_y$. Find the work done in moving a point charge $Q = -20\mu$C
   (a) From the origin to $(4, 0, 0)$ m.
   (b) From $(4, 0, 0)$ m to $(4, 2, 0)$ m.
   (c) From $(4, 2, 0)$ m to $(0, 0, 0)$ m.

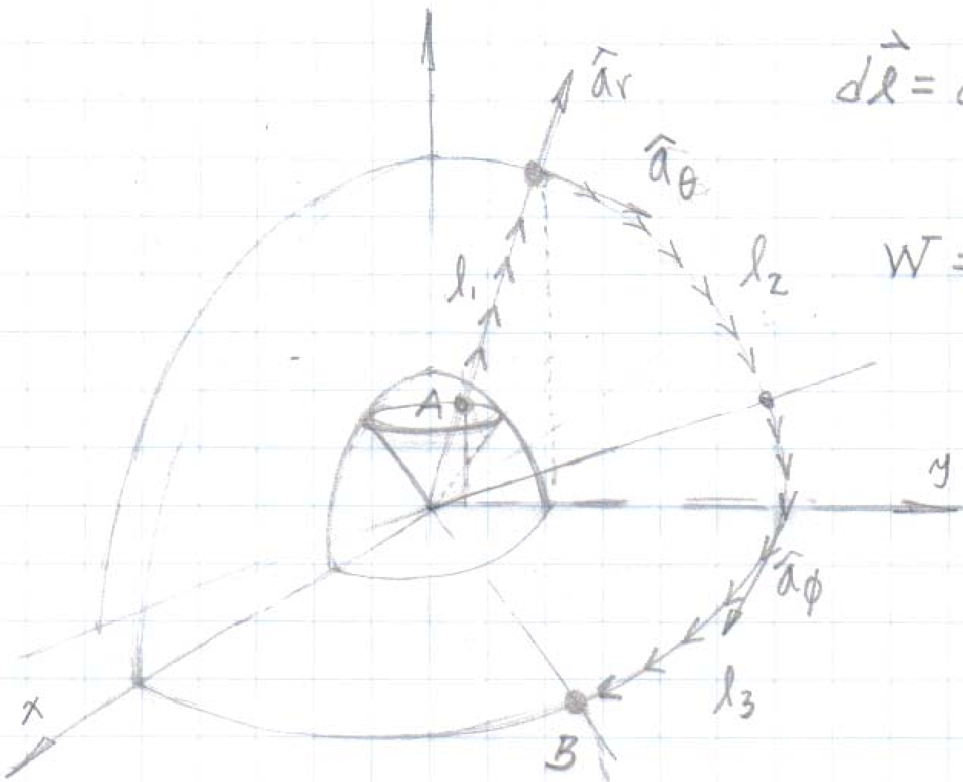
2. Two point charges -4μC and 5μC are located at $(2, -1, 3)$ and $(0, 4, -2)$, respectively. Find the potential at $(1, 0, 1)$.

3. A total charge of $40/3$ nC is uniformly distributed in the form of a circular disk of radius 2m. find the potential due to the charge at point of the axis, 2m from the disk.

4. A point charge of 5 nC is located at $(-3, 4, 0)$, while line $y = 1$, $z = 1$ carries uniform charge 2 nC/m.
   (a) If $V = 0$ V at $O(0, 0, 0)$, find V at $A(5, 0, 1)$.
   (b) If $V = 100$ V at $B(1, 2, 1)$, find V at $C(-2, 5, 3)$.
   (c) If $V = -5$ V at O, find V_{BC}.

Problems

1. Given the potential $V = \frac{10}{r^3}\sin\theta\cos\phi$,
   (a) Find the electric flux density $\vec{D}$ at $(2, \pi/2, 0)$.
   (b) Calculate de work done in moving a 10 μC charge from point $A(1, 30^0, 120^0)$ to $B(4, 90^0, 60^0)$.



$$d\vec{l} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$$

$$W = -Q\int_A^B \vec{E}\cdot d\vec{l}$$

$$\vec{E} = -\nabla V$$
$$= -\frac{10}{r^3}\left[(-2\sin\theta\cos\phi)\,\hat{a}_r + (\cos\theta\cos\phi)\,\hat{a}_\theta - \sin\phi\,\hat{a}_\phi\right]$$

Energy Density in Electrostatic Fields

Energy Density in Electrostatic Fields

- To determine the energy present in a assembly of charges, we must first determine the amount of work necessary to assemble them.

- Suppose we wish to position three point charge Q_1, Q_2 and Q_3 in an initially empty space shown shaded in Figure 1.

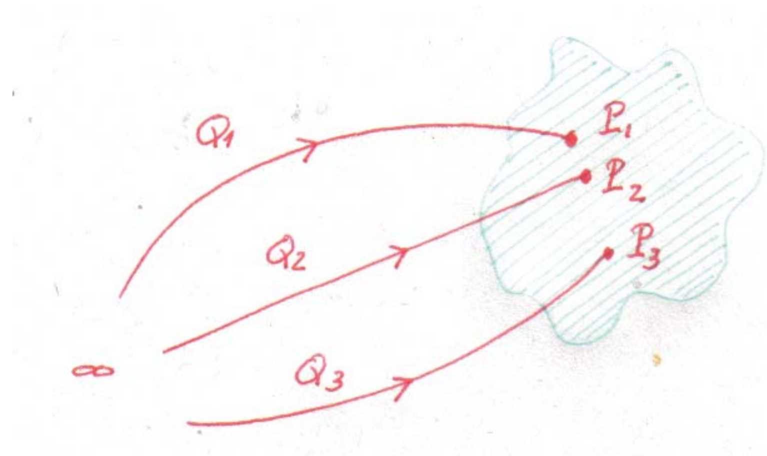


Figure 1 Assembling of charges.

- No work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field.

$$W = -Q_1 \int_A^B \vec{E} \cdot d\vec{\ell} = 0 \tag{1}$$

Energy Density in Electrostatic Fields

- The work to transferring Q_2 from infinity to P_2 is equal to the product of Q_2 and the potential V_{21} at P_2 due to Q_1.

- Similarly, the work done in positioning Q_3 at P_3 is equal to $Q_3(V_{32} + V_{31})$, where V_{32} and V_{31} are the potentials at P_3 due Q_2 and Q_1, respectively.

- Hence the total work done in positioning the three charges is

$$\begin{aligned} W_E &= W_1 + W_2 + W_3 \\ &= 0 + Q_2V_{21} + Q_3(V_{31} + V_{32}) \end{aligned} \tag{2}$$

- If the charges were positioned in reverse order,

$$\begin{aligned} W_E &= W_3 + W_2 + W_1 \\ &= 0 + Q_2V_{23} + Q_1(V_{12} + V_{13}) \end{aligned} \tag{3}$$

where V_{23} is the potential at P_2 due to Q_3, V_{12} and V_{13} are, respectively, the potential at P_1 due to Q_2 and Q_3.

Energy Density in Electrostatic Fields

- Adding eqs. (2) and (3) gives

$$\begin{aligned} 2W_E &= Q_1(V_{12} + V_{13}) + Q_2(V_{21} + V_{23}) + Q_3(V_{31} + V_{32}) \\ &= Q_1V_1 + Q_2V_2 + Q_3V_3 \end{aligned} \tag{4}$$

- or

$$W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3) \tag{5}$$

where V_1, V_2 and V_3 are the total potentials at P_1,P_2 and P_3, respectively.

- In general, if there are n point charges, eq.(5) become

$$W_E = \frac{1}{2}\sum_{k=1}^{n} Q_kV_k \quad (\text{ in joules}) \tag{6}$$

Energy Density in Electrostatic Fields

- If, instead of point charges, the region has a continuous charge distribution, the summation in eq.(6) becomes integration; that is,

$$W_E = \frac{1}{2}\int_L \rho_\ell V\, d\ell \quad \text{(line chrage)} \tag{7}$$

$$W_E = \frac{1}{2}\int_S \rho_S V\, dS \quad \text{surface chrage} \tag{8}$$

$$W_E = \frac{1}{2}\int_\nu \rho_\nu V\, d\nu \quad \text{volume chrage} \tag{9}$$

- Since $\rho_v = \nabla \cdot \vec{D}$, eq.(9) can be further developed to yield

$$W_E = \frac{1}{2}\int_\nu (\nabla \cdot \vec{D}) V\, d\nu \quad \text{volume chrage} \tag{10}$$

- But for any vector $\vec{A}$ and escalar V, the identity $\nabla \cdot V\vec{A} = \vec{A} \cdot \nabla V + V(\nabla \cdot \vec{A})$ or

$$(\nabla \cdot \vec{A})V = \nabla \cdot V\vec{A} - \vec{A} \cdot \nabla V \tag{11}$$

holds.

Energy Density in Electrostatic Fields

- Applying the identity in eq.(11) to (10), we get

$$W_E = \frac{1}{2}\int_{\nu}(\nabla \cdot \vec{D})V\,d\nu = \frac{1}{2}\int_{\nu}(\nabla \cdot V\vec{D})\,d\nu - \frac{1}{2}\int_{\nu}(\vec{D}\cdot\nabla V)\,d\nu \tag{12}$$

- But applying divergence theorem to the first term on the right-hand side of this equation, we have

$$W_E = \frac{1}{2}\oint_{S}(V\vec{D})\cdot d\vec{S} - \frac{1}{2}\int_{\nu}(\vec{D}\cdot\nabla V)\,d\nu \tag{13}$$

- Hence, eq.(13) reduce to

$$W_E = -\frac{1}{2}\int_{\nu}(\vec{D}\cdot\nabla V)\,d\nu = \frac{1}{2}\int_{\nu}(\vec{D}\cdot\vec{E})\,d\nu \tag{14}$$

- and since $\vec{E} = -\nabla V$ and $\vec{D} = \varepsilon_0\vec{E}$

$$W_E = \frac{1}{2}\int_{\nu}(\vec{D}\cdot\vec{E})\,d\nu = \frac{1}{2}\int_{\nu}\varepsilon_0 E^2\,d\nu \tag{15}$$

Energy Density in Electrostatic Fields

- From this, we can define electrostatic energy density w_E (in J/m^3) as

$$w_E = \frac{dW_E}{d\nu} = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\varepsilon_0 E^2 = \frac{D^2}{2\varepsilon_0} \tag{16}$$

- so eq.(14) may be written as

$$W_E = \int_\nu w_E d\nu \tag{17}$$

MAGNETOSTATICS

Magnetostatic

- In the last sections, we limited our discussions to static electric fields characterized by $\vec{E}$ or $\vec{D}$.
- We now focus our attention on static magnetic fields, which are characterized by $\vec{H}$ or $\vec{B}$.
- There are similarities and dissimilarities between electric and magnetic fields.
- As $\vec{E}$ and $\vec{D}$ are related according to $\vec{D} = \varepsilon\vec{E}$ for linear, isotropic material space, $\vec{H}$ and $\vec{B}$ are realted according to $\vec{B} = \mu\vec{H}$.

Magnetostatic

- Analogy between Electric ($\vec{D} = \varepsilon\vec{E}$) and Magnetic Fields ($\vec{B} = \mu\vec{H}$).

Term	Electric	Magnetic
Basic laws	$\vec{F} = \frac{Q_1 Q_2}{4\pi\varepsilon r^2}\,\hat{a}_r$	$d\vec{B} = \frac{\mu I d\vec{\ell}\times\hat{a}_r}{4\pi R^2}$
	$\oint \vec{D}\cdot d\vec{S} = Q_{enc}$	$\oint \vec{H}\cdot d\vec{\ell} = I_{enc}$
Force law	$\vec{F} = Q\vec{E}$	$\vec{F} = Q\vec{u}\times\vec{B}$
Source element	dQ	$Q\vec{u} = I d\vec{\ell}$
Field intensity	$E = \frac{V}{\ell}$ (V/m)	$H = \frac{I}{\ell}$ (A/m)
Flux density	$\vec{D} = \frac{\psi}{S}$ (C/m^2)	$\vec{B} = \frac{\psi}{S}$ (Wb/m^2)
Relationships between fields	$\vec{D} = \varepsilon\vec{E}$	$\vec{B} = \mu\vec{H}$
Potentials	$\vec{E} = -\nabla V$	$\vec{H} = -\nabla V_m$ ($\vec{J} = 0$)
	$V = \int \frac{\rho_\ell\, d\ell}{4\pi\varepsilon r}$	$A = \int \frac{\mu I\, d\ell}{4\pi R}$
Flux	$\psi = \int \vec{D}\cdot d\vec{S}$	$\psi = \int \vec{B}\cdot d\vec{S}$
	$\psi = Q = CV$	$\psi = LI$
	$I = C\frac{dV}{dt}$	$V = L\frac{dI}{dt}$
Energy density	$W_E = \frac{1}{2}\vec{D}\cdot\vec{E}$	$W_m = \frac{1}{2}\vec{B}\cdot\vec{H}$
Poisson's equation	$\nabla^2 V = \frac{\rho_V}{\varepsilon}$	$\nabla^2 A = -\mu\vec{J}$

Magnetostatic

- A definite link between electric and magnetic fields was established by Hans Christian Oersted (1777-1851) in 1820, Danish professor of physics.
- As we have noticed, an electrostatic field is produced by static or stationary charges.
- If the charges are moving with constant velocity, a static magnetic (or magnetostatic) field is produced.
- A magnetostatic field is produced by a constant current flow (or direct current).
- This current flow may be due to magnetization currents as in permanent magnets, electron-beam currents as in vacuum tubes, or condition currents as in current-carrying wires.
- In this section, we consider magnetic fields in free space due to direct current. Motors, transformers, microphones, compasses, telephone bell ringers, television focussing controls, advertising displays, magnetically levitated high-speed vehicles, memory store, magnetic separators, and so on, which play an important role in our everyday life, could not have been developed without an understanding of magnetic phenomena.

Magnetostatic

- There are two major laws governing magnetostatic fields: (1) Biot-Savart's law, and (2) Ampère's law.
- Like Coulomb's law, Biot-Savart's law is the general law of magnetostatics.
- Just as Gauss's law is a special case of Coulomb's law, Ampère's law is a special case of Biot-Savart's law and is easily applied in problems involving symmetrical current distribution.

Biot-Savart's Law

- **Biot-Savart law** states that the differential magnetic field intensity dH produced at a point P, as shown in Figure 1, by the differential current element $Id\ell$ is proportional to de product $Id\ell$ and the sine of the angle α between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element.

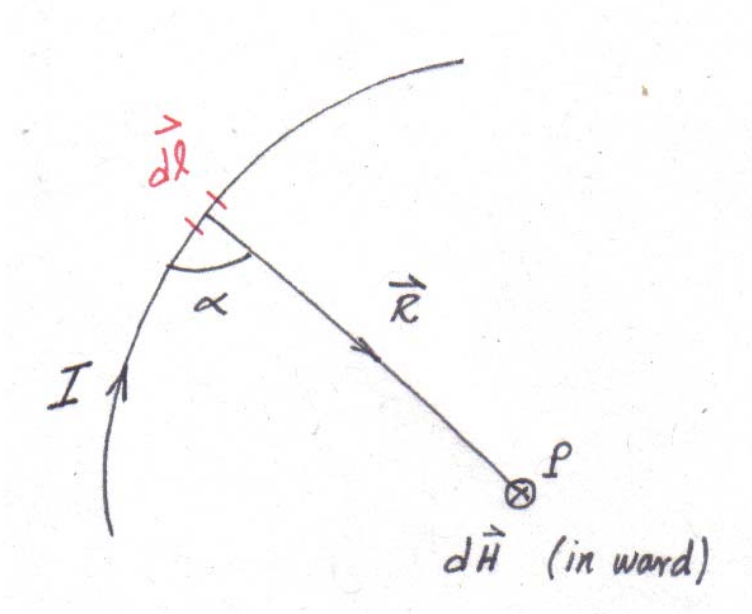


Figure 1 Magnetic field $d\vec{H}$ at P due to current element $Id\vec{\ell}$.

- That is,

$$dH = \frac{Id\ell \sin\alpha}{4\pi R^2} \tag{1}$$

Biot-Savart's Law

- From the definition of cross product, it is easy to notice that eq.(1) is better put in vector form as

$$d\vec{H} = \frac{I d\vec{\ell} \times \hat{a}_R}{4\pi R^2} = \frac{I d\vec{\ell} \times \vec{R}}{4\pi R^3} \qquad (2)$$

where $R = |\vec{R}|$ and $\hat{a}_R = \vec{R}/R$; $\vec{R}$ and $d\vec{\ell}$ are illustrated in Figure 1.

- The direction of $d\vec{H}$ can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current and the right-hand fingers encircling the wire in the direction of $d\vec{H}$ as shown in Figure 2.

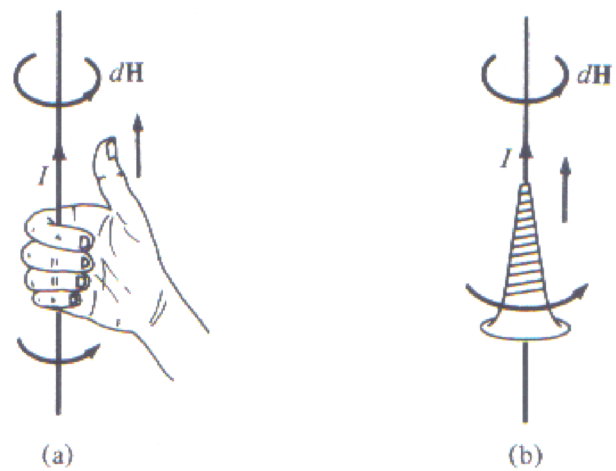


Figure 2 Determining the direction of $d\vec{H}$ using (a) the right-hand rule or (b) the right-handed-screw rule.

Biot-Savart's Law

- Just as we can have different charge configurations, we can have different current distributions: line current, surface current, and volume current as shown in Figure 3.

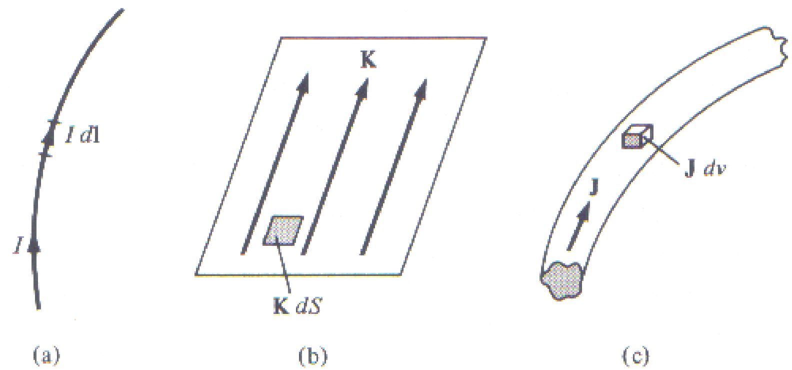


Figure 3 Current distributions: (a) line current, (b) surface current, (c) volume current.

- If we define $\vec{K}$ as the surface current density in A/m and $\vec{J}$ as the volume current density in A/m^2, the source element are related as

$$I d\vec{\ell} = \vec{K} dS = \vec{J} dV \tag{3}$$

Biot-Savart's Law

- Thus in terms of the distributed current source, the Biot-Savart law as in eq.(2) become

$$\vec{H} = \int_L \frac{I d\vec{\ell} \times \hat{a}_R}{4\pi R^2} \quad \text{line current} \tag{4}$$

$$\vec{H} = \int_S \frac{\vec{K} dS \times \hat{a}_R}{4\pi R^2} \quad \text{surface current} \tag{5}$$

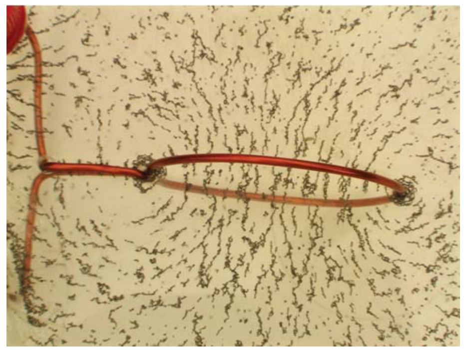
$$\vec{H} = \int_V \frac{\vec{J} dV \times \hat{a}_R}{4\pi R^2} \quad \text{volume current} \tag{6}$$

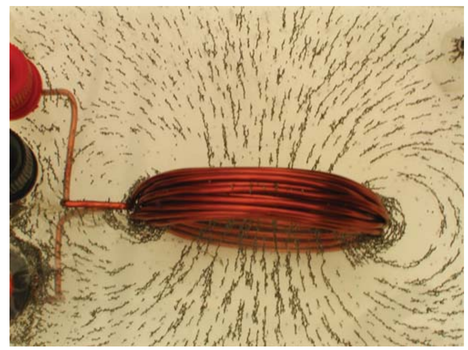
where $\hat{a}_R$ is a unit vector pointing from the differential element of current to the point of interes.

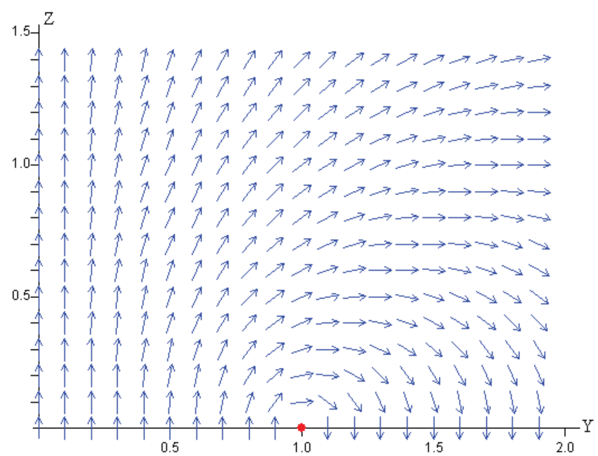
Biot-Savart's Law

- Circular lop of radius ρ, carries a direct current I

$$\vec{H} = \frac{I\rho^2}{2(\rho^2 + z^2)^{3/2}}\,\widehat{a}_z \tag{7}$$





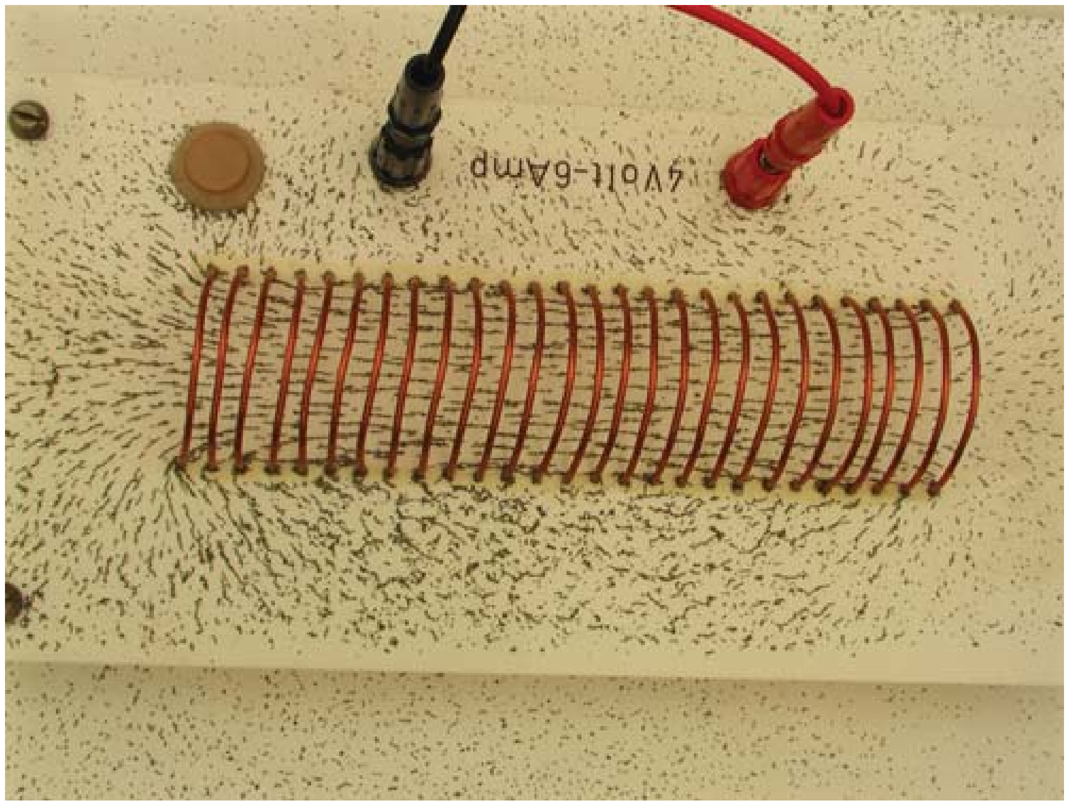


$$z = 0, \quad \vec{H} = \frac{I}{2\rho}\,\widehat{a}_z \tag{8}$$

Biot-Savart's Law

- A solenoid of length ℓ, radius a and N turns of wire carries current I.

$$\vec{H} = \frac{IN}{2\sqrt{a^2 + \frac{\ell^2}{4}}}\,\hat{a}_z \tag{9}$$



$$\ell >> a, \quad \vec{H} = \frac{IN}{\ell}\,\hat{a}_z \tag{10}$$

AMPÈRE'S CIRCUIT LAW

Ampère's circuit law

- States that the integral of $\vec{H}$ around a *closed* path is the same as the net current I_{enc} enclosed by the path.

- In other words, the circulation of $\vec{H}$ equals I_{enc}: that is,

$$\oint \vec{H} \cdot d\vec{\ell} = I_{enc} \tag{1}$$

- Ampèr's law is similar to Gauss's law, since Ampèr's law is easily applied to determine $\vec{H}$ when the current distribution is symmetrical.

- By applying Stoke's theorem to the left-hand side of eq. (1), we obtain

$$I_{enc} = \oint_{\zeta} \vec{H} \cdot d\vec{\ell} = \int_{S} (\nabla \times \vec{H}) \cdot d\vec{S} \tag{2}$$

Ampère's circuit law

- But

$$I_{enc} = \int_S \vec{J} \cdot d\vec{S}, \tag{3}$$

  where $\vec{J}$ is *current density*.

- Comparing the surface integrals in eqs. (2) and (3) clearly reveals that

$$\nabla \times \vec{H} = \vec{J} \tag{4}$$

- This is the third Maxwell's equation to be derived; it is essentially Ampère's law in differential (or point) form, whereas eq. (1) is the integral form.

- From eq. (4), we should observe that $\nabla \times \vec{H} = \vec{J} \neq 0$; that is a magnetostatic field is not conservative.

MAGNETIC FLUX DENSITY

Magnetic Flux Density

- The magnetic flux density $\vec{B}$ is similar to the electric flux density $\vec{D}$.

- As $\vec{D} = \varepsilon_o \vec{E}$ in free space, the magnetics flux density $\vec{B}$ is related to the magnetic field intensity $\vec{H}$ according to

$$\vec{B} = \mu_o \vec{H} \tag{1}$$

where μ_o is a constant known as the *permeability of free space*. The constant is in henrys por meter (H/m) and has the value of

$$\mu_o = 4\pi \times 10^{-7} \text{ H/m} \tag{2}$$

- The magnetic flux through a surface S is given by

$$\psi = \int_S \vec{B} \cdot d\vec{S} \tag{3}$$

where the magnetic flux ψ is in Webers (Wb) and the magnetic flux density is in Webers per square meter (Wb/m^2) or Tesla (T).

Magnetic Flux Density

- $$\psi = \int_S \vec{B} \cdot d\vec{S}$$

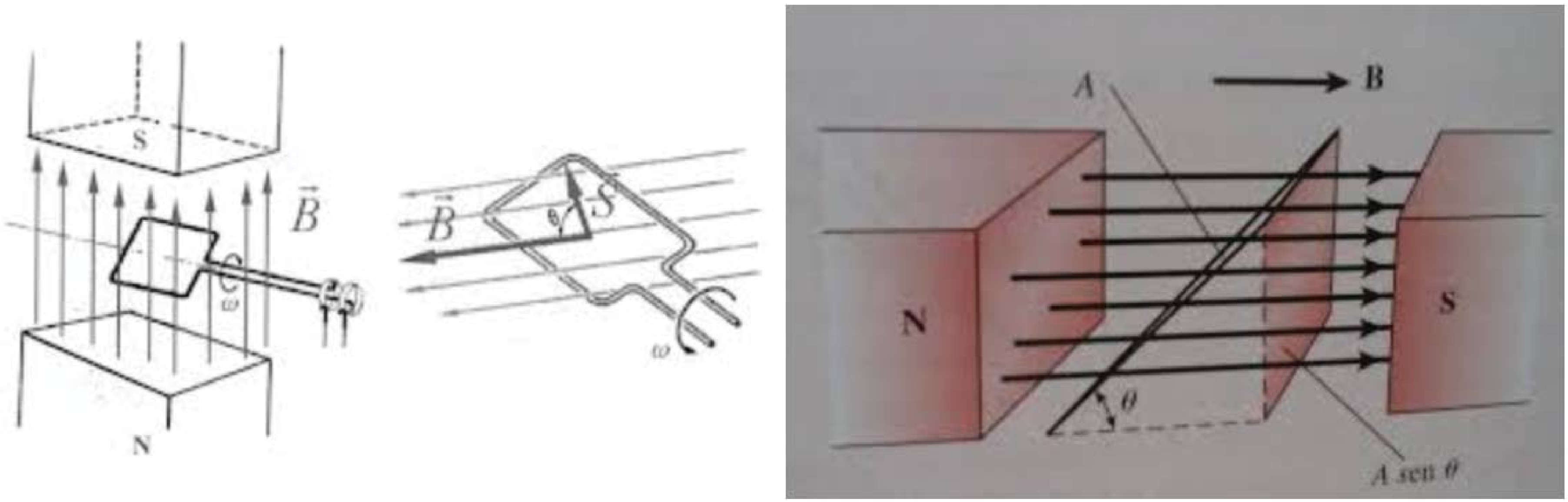


Figure 1 Magnetic flux

Magnetic Flux Density

- A magnetic flux line is a path to which $\vec{B}$ is tangential a every point on the line.

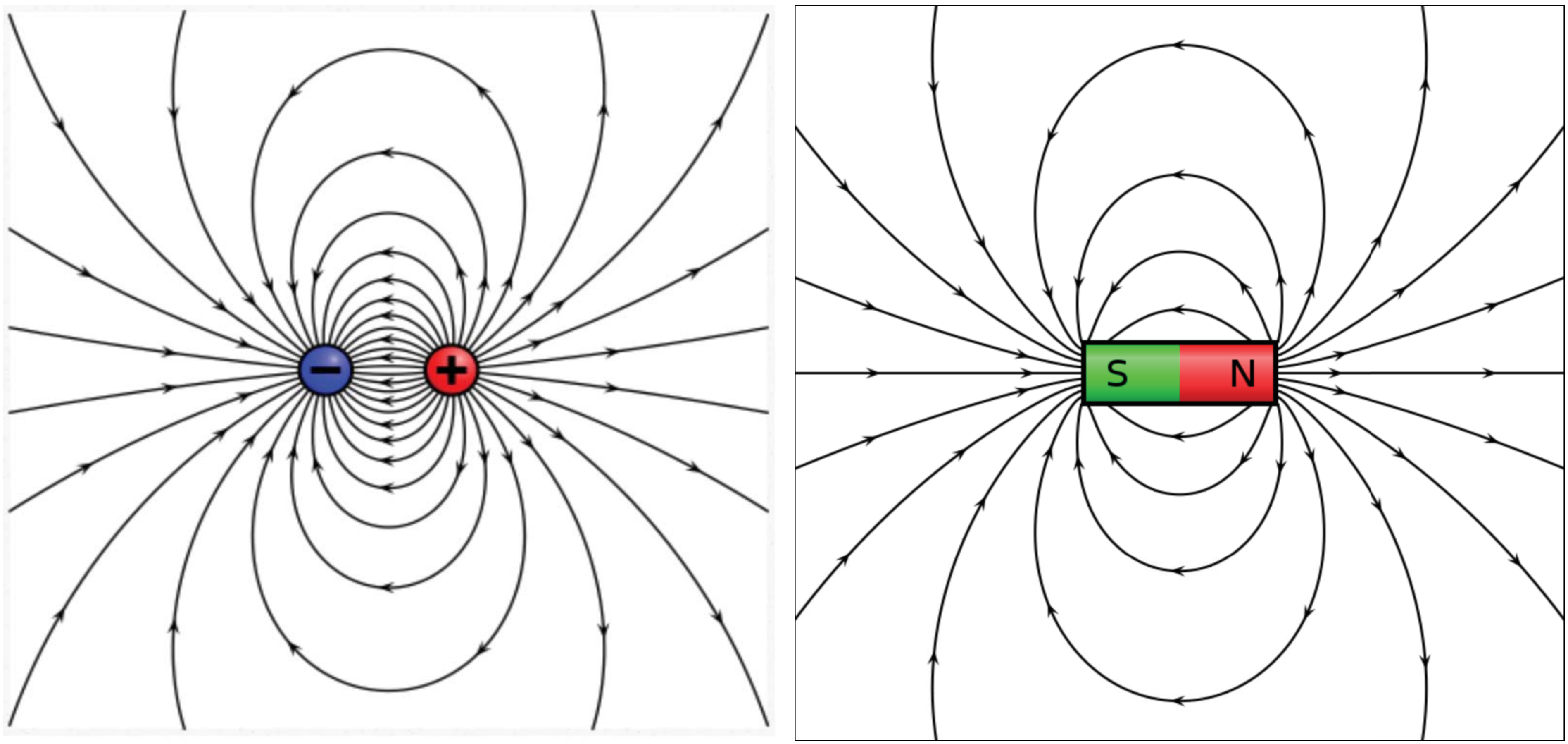


Figure 2: a) Electric Dipole and b) Magnetic Dipole

Magnetic Flux Density

- In an electrostatic field, the flux passing through a closed surface is te same as the charge enclosed; that is $\psi = \oint_S \vec{E} \cdot d\vec{S} = Q$.
- Thus it is possible to have an isolate electric charge as shown in Figure 3a).
- Which also reveals that electric flux lines are not necessarily closed.

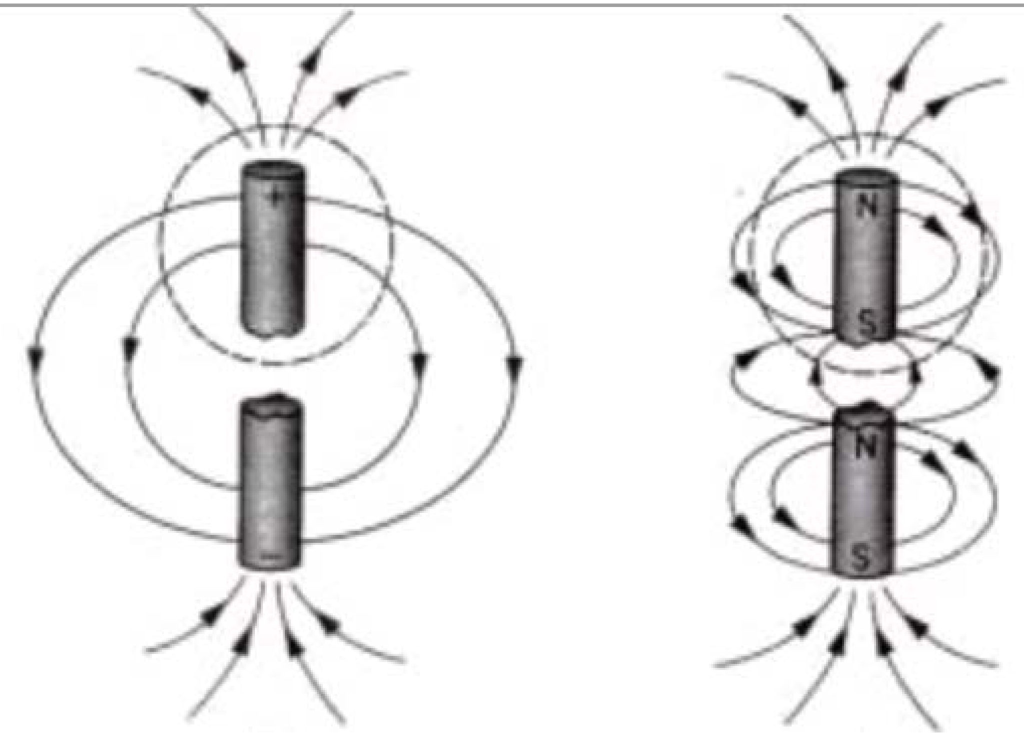


Figure 3: a) Isolate electric charge and b) Magnetic Dipole

Magnetic Flux Density

- Unlike electric flux flied, magnetic flux lines always close upon themselves as in Figure 3b).
- This is because it is not possible to have isolated magnetic poles (or magnetic charges).
- And isolated magnetic charge does not exist. Thus the total flux through a closed surface in a magnetic field must be zero; that is, $\psi = \oint_S \vec{B} \cdot d\vec{S} = 0$.

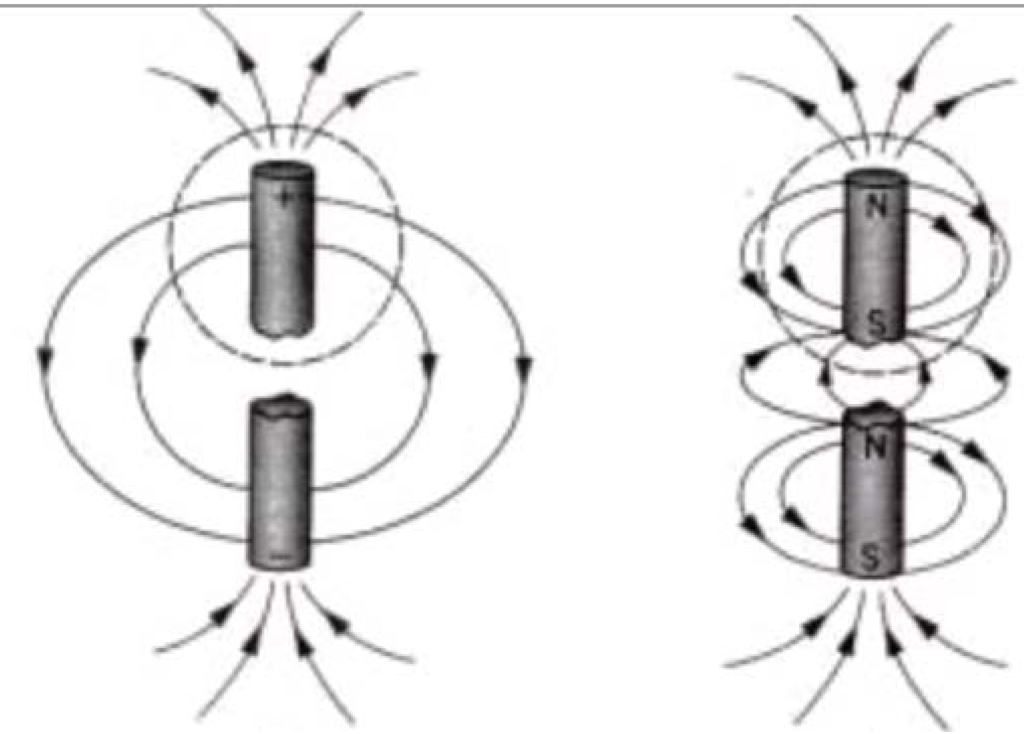


Figure 3: a) Isolate electric charge and b) Magnetic Dipole

Magnetic Flux Density

- $$\psi = \oint_S \vec{B} \cdot d\vec{S} = 0 \tag{4}$$

- This equation is referred to as the law of conservation of magnetic flux or Gauss's law for magnetostatic fields.

- By applying the divergence theorem, we obtain

  $$\psi = \oint_S \vec{B} \cdot d\vec{S} = \int_V \nabla \cdot \vec{B} dV = 0 \tag{5}$$

  or

  $$\nabla \cdot \vec{B} = 0 \tag{6}$$

MAGNETIC SCALAR AND VECTOR POTENTIALS

Magnetic scalar and vector potentials

- The magnetic flux through a surface S is given by

$$\psi = \int_S \vec{B} \cdot d\vec{S} \tag{11}$$

  where the magnetic flux ψ is in Webers (Wb) and the magnetic flux density is in Webers per square meter (Wb/m^2) or Tesla (T).

- The *vector magnetic potential* $\vec{A}$ (in Wb/m) such that

$$\vec{B} = \nabla \times \vec{A} \tag{12}$$

- By substituting eq. (12) in to eq.(11) and applying Stokes's theorem, we obtain

$$\psi = \int_S \vec{B} \cdot d\vec{S} = \int_S (\nabla \times \vec{A}) \cdot d\vec{S} = \oint_\ell \vec{A} \cdot d\ell \tag{13}$$

Magnetic scalar and vector potentials

- We can define

$$\vec{A} = \int_{\ell} \frac{\mu_o I \, d\vec{\ell}}{4\pi R} \quad \text{for line charge} \tag{8}$$

$$\vec{A} = \int_{S} \frac{\mu_o \vec{K} \, dS}{4\pi R} \quad \text{for surface current} \tag{9}$$

$$\vec{A} = \int_{\nu} \frac{\mu_o \vec{J} \, d\nu}{4\pi R} \quad \text{for volume current} \tag{10}$$

Magnetic scalar and vector potentials

- We known that for magnetostatic field $\nabla \cdot \vec{B} = 0$.

- To satisfy last eq. and eq. (2) simultaneously, we can define the *vector magnetic potential* $\vec{A}$ (in Wb/m) such that

$$\vec{B} = \nabla \times \vec{A} \tag{6}$$

- Just as we define

$$V = \int \frac{dQ}{4\pi\varepsilon_0 r} \tag{7}$$

Magnetic scalar and vector potentials

- Just as $\vec{E} = -\nabla V$, we define the *magnetic scalar potential* V_m (in amperes) as related to $\vec{H}$ according to

$$\vec{H} = -\nabla V_m \qquad \text{if} \qquad \vec{J} = 0 \tag{3}$$

- The condition attached to this equation is importan. Combining eq. (3) and $\nabla \times \vec{H} = \vec{J}$ give

$$\vec{J} = \nabla \times \vec{H} = \nabla \times (-\nabla V_m) = 0 \tag{4}$$

since V_m must satisfy the condition in eq. (1).

- Thus the magnetic scalar potential V_m is only defined in a region where $\vec{J} = 0$ as eq. (3).

- We should also note that V_m satisfies Laplace's equation just V does for electrostatic field; hence,

$$\nabla^2 V_m = 0, \qquad (\vec{J} = 0) \tag{5}$$

Magnetic scalar and vector potentials

- We recall that some electrostatics field problems were simplified by relating the electric potential V to the electric field intensity $\vec{E}$ ($\vec{E} = -\nabla V$).
- Similarly, we can define a potential associated with magnetostatic field $\vec{B}$.
- In fact, the magnetic potential could be scalar V_m or vector $\vec{A}$.
- To define V_m and $\vec{A}$ involves recalling two important identities

$$\nabla \times (\nabla V) = 0 \tag{1}$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \tag{2}$$

which must always hold for any scalar field V and vector field $\vec{A}$.

Maxwell's equations for statics fields

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot \vec{D} = \rho_\nu$	$\oint_S \vec{D} \cdot d\vec{S} = \int_\nu \rho_\nu d\nu$	Gauss's law
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{S} = 0$	Nonexistence of magnetic monopole
$\nabla \times \vec{E} = 0$	$\oint_\ell \vec{E} \cdot d\vec{\ell} = 0$	Conservative nature of electrostatic field
$\nabla \times \vec{H} = \vec{J}$	$\oint_\ell \vec{H} \cdot d\vec{\ell} = \oint_S \vec{J} \cdot d\vec{S}$	Ampère's law

Gauss's law for electric fields

- The integral form of Gauss's law

$$\oint_S \vec{E} \cdot \hat{n} da = \frac{q_{enc}}{\varepsilon_0}$$

The left side of this equation is no more than a mathematical description of the electric flux - the number of electric field lines - passing through a closed surface S, whereas the right side is the total amount of charge contained within that surface divided by a constant called the permittivity of free space.

★Electric charge produces an electric field, and the flux of that field passing through any closed surface is proportional to the total charge contained within that surface.

- The differential form of Gauss's law

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

The left side of this equation is a mathematical description of the divergence of the electric field - the tendency of the field to "flow" away from a specified location - and the right side is the electric charge density divided by the permittivity of free space.

★ The electric field produced by electric charge diverges from positive charge and converges upon negative charge.

Gauss's law for magnetic fields

- The integral form of Gauss's law

$$\oint_S \vec{B} \cdot \widehat{n} da = 0$$

In this case, Gauss's law refers to magnetic flux - the number of magnetic field lines - passing through a closed surface S. The right side is identically zero.

★ The total magnetic flux passing through any closed surface is zero.

- The differential form of Gauss's law

$$\nabla \cdot \vec{B} = 0$$

The left side of this equation is a mathematical description of the divergence of the magnetic field - the tendency of the magnetic field to "flow" more strongly away from a point than toward it - while the the right side is simply zero.

★ The divergence of the magnetic field at any point is zero.

Faraday's law

- The integral form of Faraday's law

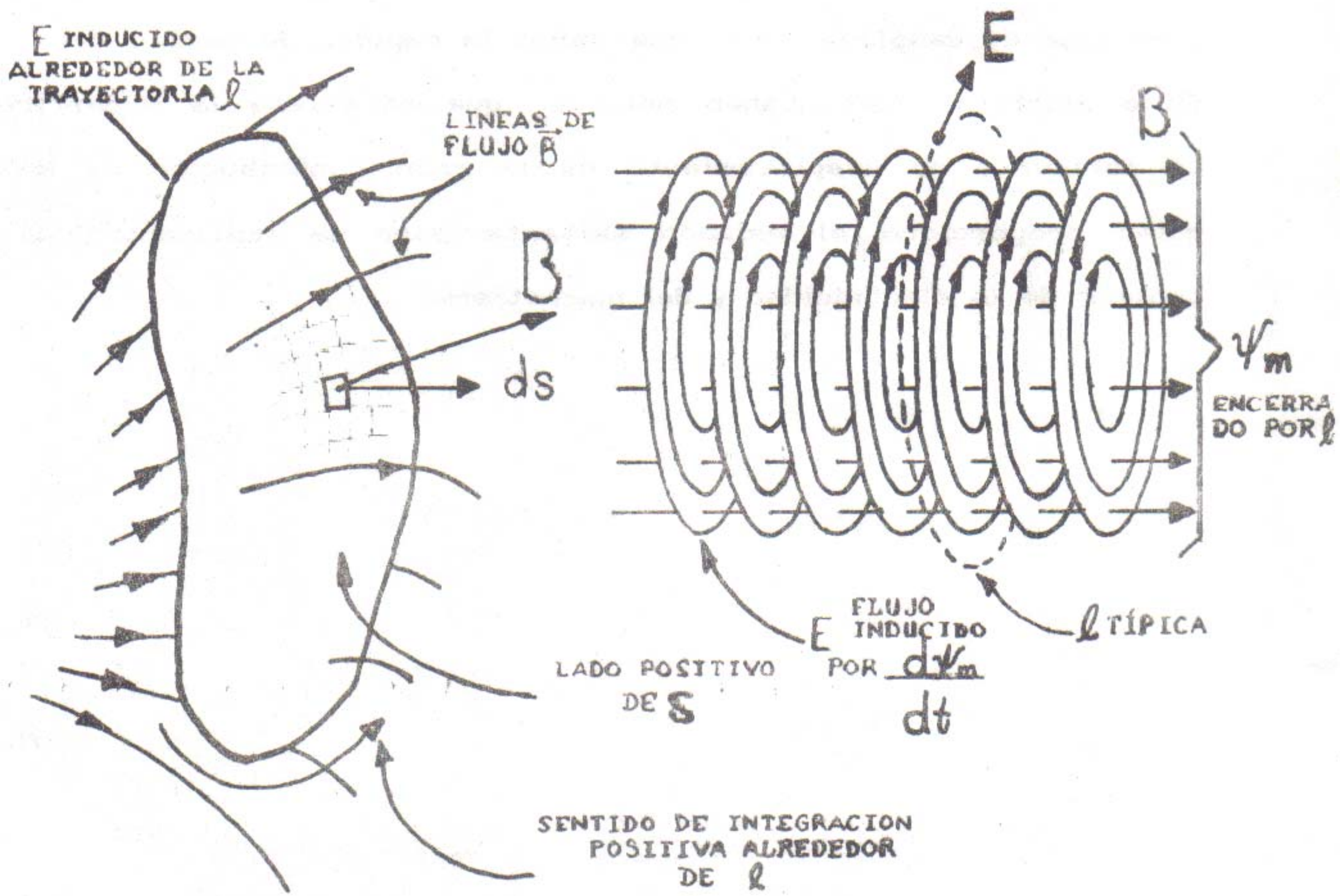
$$\oint_C \vec{E} \cdot \vec{dl} = -\frac{d}{dt}\int_S \vec{B} \cdot \hat{n} da$$

$$\text{emf} = -\frac{d}{dt}\int_S \vec{B} \cdot \hat{n} da \qquad \text{(Flux rule)}$$

★ Changing magnetic flux through a surface induce an emf - motional electromotive force - in any boundary path of that surface, and changing magnetic field induces a circulating electric field.

$$\oint_C \vec{E} \cdot \vec{dl} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} da \quad \text{(alternate form)}$$

• The integral form of Faraday's law



- The differential form of Faraday's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The left side of this equation is a mathematical description of the curl of the electric field - the tendency of the field lines to circulate around a point. The right side represents rate of change of the magnetic field over time.

★ A circulating electric field is produced by a magnetic field that changes with time.

Ampere-Maxwell law

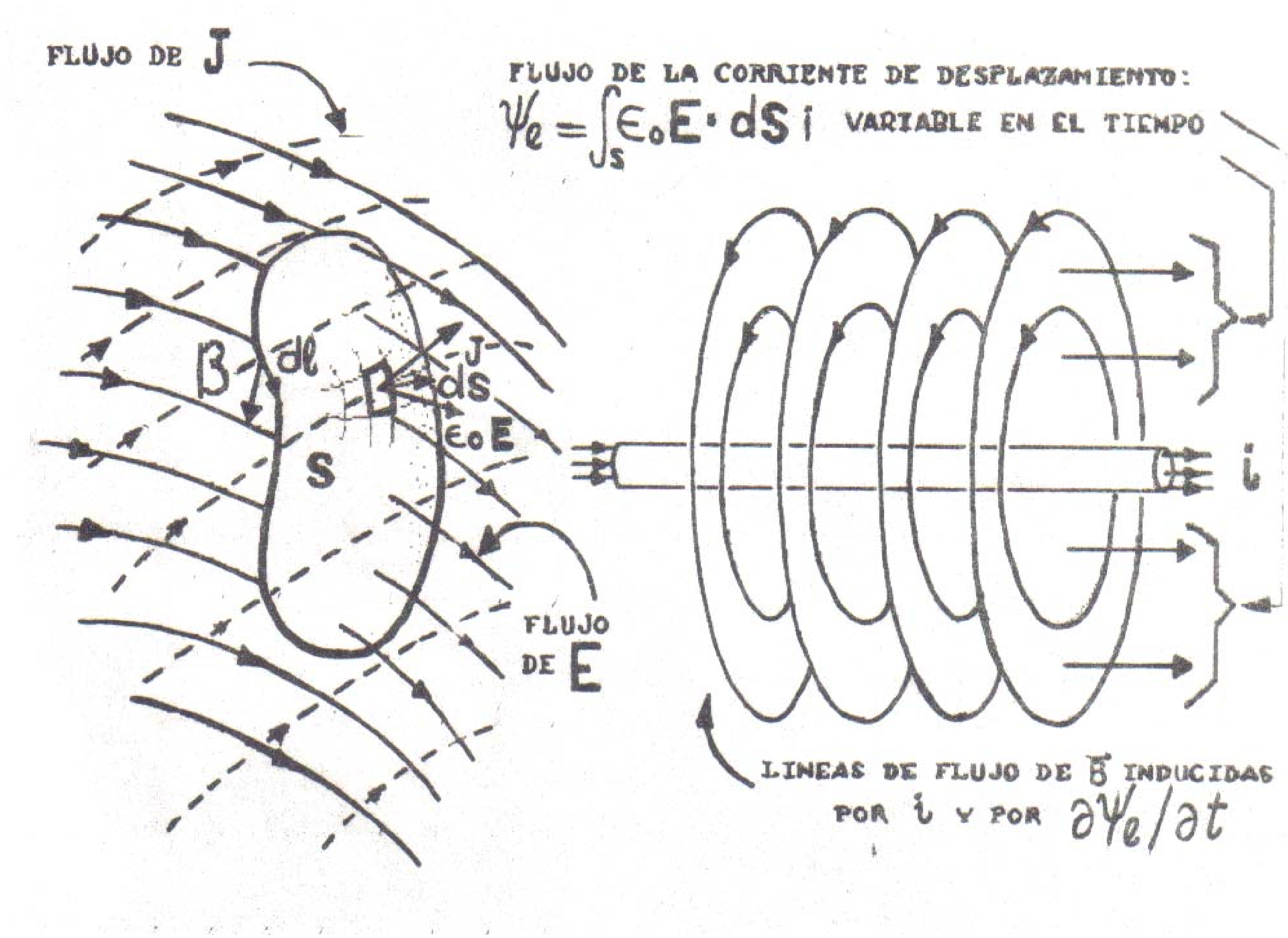
- The integral form of the Ampere-Maxwell law

$$\oint_C \vec{B} \cdot \vec{dl} = \mu_0 (I_{enc} + \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n} da)$$

The left side of this equation is a mathematical description of the circulation of the magnetic field around a closed path C. The right side include two sources for the magnetic field; a steady conduction current and a changing electric flux through any surface S bounded by path C.

★ An electric current or a changing electric flux through a surface produces a circulating magnetic field around any path that bounds that surface.

- The integral form of the Ampere-Maxwell law



- The differential form of the Ampere-Maxwell law

$$\nabla \times \vec{B} = \mu_0(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t})$$

The left side of this equation is mathematical description of the curl of the magnetic field - the tendency of the field lines to circulate around a point. The two terms on the right side represent the electric current density and the time of change of the electric field.

★ A circulating magnetic field is produced by an electric current and by an electric field that changes with time.

From Maxwell's Equations to the wave equation

- **Gauss's law for electric fields:**

$$\oint_S \vec{E} \cdot \hat{n} da = \frac{q_{enc}}{\varepsilon_0} \qquad \xrightarrow[theorem]{Divergence} \qquad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

- **Gauss's law for magnetic fields:**

$$\oint_S \vec{B} \cdot \hat{n} da = 0 \qquad \xrightarrow[theorem]{Divergence} \qquad \nabla \cdot \vec{B} = 0$$

- **Faraday's law:**

$$\oint_C \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \int_S \vec{B} \cdot \hat{n} da \qquad \xrightarrow[theorem]{Stokes'} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- **Ampere-Maxwell law:**

$$\oint_C \vec{B} \cdot \vec{dl} = \mu_0 (I_{enc} + \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n} da) \qquad \xrightarrow[theorem]{Stokes'} \qquad \nabla \times \vec{B} = \mu_0 (\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t})$$

- The wave equation for electric and magnetic fields:

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

• Generalizing Amepere's Law

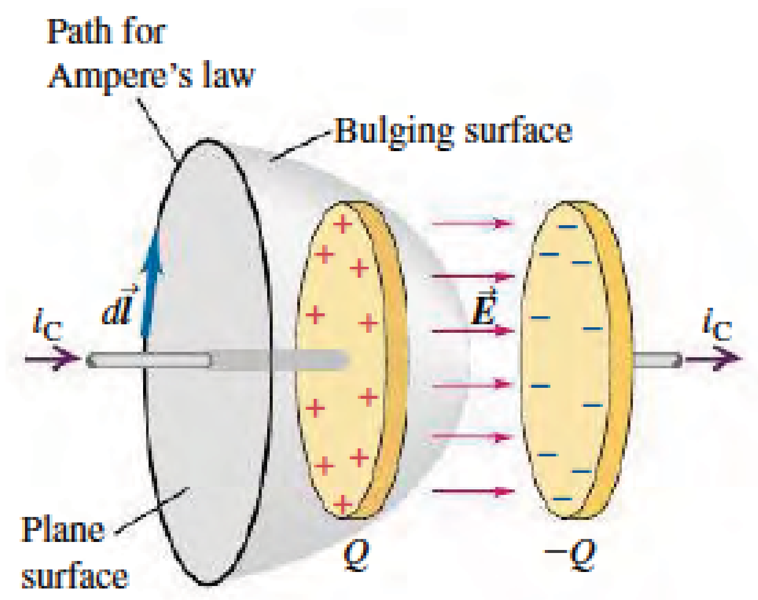
$$\oint_{\zeta} \vec{B} \cdot \vec{dl} = \mu_0 (I_{encl})$$

The problem with Ampere's law in this form is that it is *imcomplete*. To see why, let's consider the process of charging a capacitor.

Conducting wire lead current i_C into plate and out of the other; the charge Q increases, and the field $\vec{E}$ between the plates increases.

The notation i_C indicate *conduction* current to distinguish it form another kind of current we are about to encounter, called *displacement* current i_D.

Let's apply Ampere's law to the circular path shown. The integral $\oint_{\zeta} \vec{B} \cdot \vec{dl}$ around this path equals $\mu_0 (I_{encl})$.

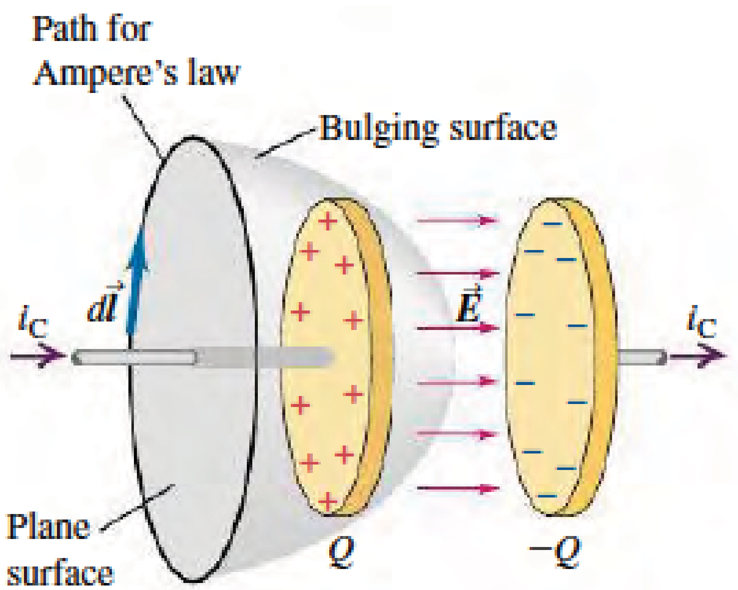


- Generalizing Amepere's Law

For the plate circular area bounded by the circle, I_{encl} is just the current i_C in the left conductor.

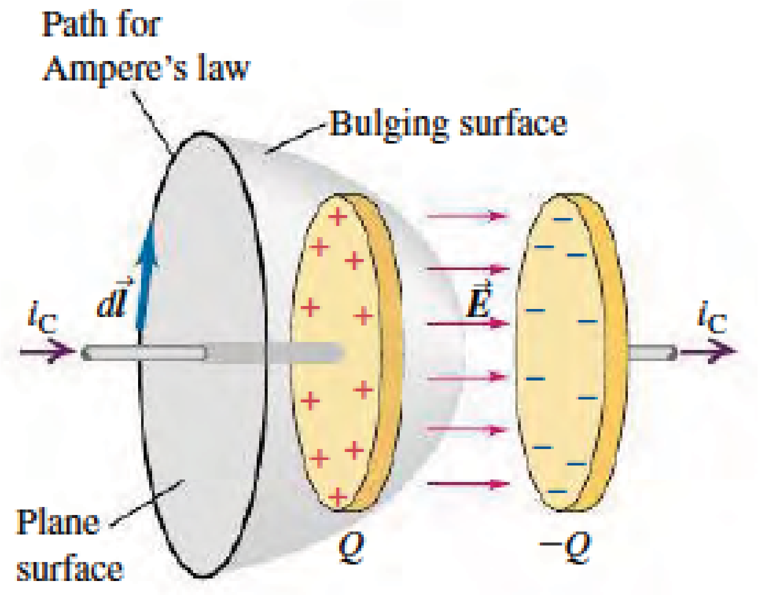
But the surface that bulges out to the right is bounded by the same circle, and the current through that surface is zero.

So $\oint_\zeta \vec{B}\cdot d\vec{l}$ is equal to $\mu_0(I_{encl})$, and the same time it is equal to zero! This is a clear contradiction. However, something else is happening on the bulged-out surface.



- Generalizing Amepere's Law

As the capacitor charges, the electric field $\vec{E}$ and the electric *flux* Φ_E through the surface are increasing. We can determine their rate of charge in terms of the charge and current. The instantaneous charge is $q = Cu$, where C is the capacitance and u is the instantaneous potential difference. For a parallel-plate capacitor $C = \epsilon_0 A/d$, where A is the plate area and d is the spacing. The potential difference u between plates is $u = Ed$, where E is the electric-field magnitude between plates. If this region is filled with a material with permittivity ϵ, we replace ϵ_0 by ϵ everywhere.



• Generalizing Amepere's Law

Substituting these expressions for C and u into $q = Cu$, we can express the capacitor charge q in terms of the electric flux $\Phi_E = EA$ through the surface:

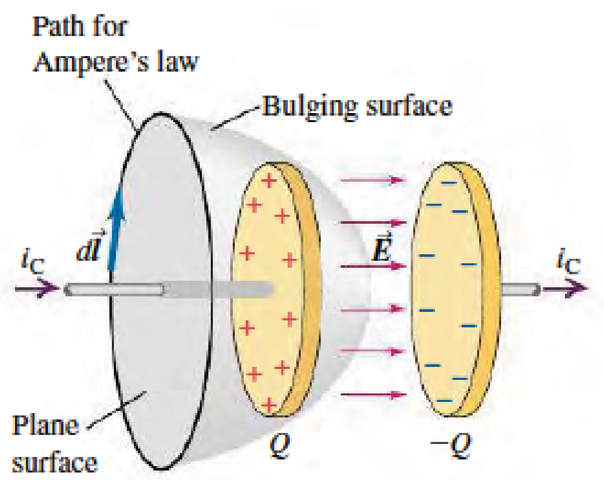
$$q = Cu = \frac{\epsilon A}{d}(Ed) = \epsilon \Phi_E \tag{1}$$

As the capacitor charges, the rate of charge of q is the conduction current, $i_C = dq/dt$. Taking the derivative of Eq.(**??**) with respect to time, we get

$$i_C = \frac{dq}{dt} = \epsilon \frac{d\Phi_E}{dt} \tag{2}$$

Stretching our imagination a bit, we invent a fictitious **displacement current** i_D in the region between the plates, defined as

$$i_D = \epsilon \frac{d\Phi_E}{dt} \tag{3}$$



Generalizing Amepere's Law

That is, we imagine that the changing flux through the curved (bulged-out) surface is equivalent, in Ampere's law, to a conduction current through that surface.

$$\oint_C \vec{B} \cdot \vec{dl} = \mu_0 (i_C + i_D) \tag{4}$$

Ampere's law in this form is obeyed no matter which surface we use in Figure. For the flat surface, i_D is zero; for the curved surface, i_C is zero; and i_C for the flat surface equals i_D for the curved surface. Equation (**??**) remains valid in a magnetic material, provided that the magnetization is proportional to external field and we replace μ_0 by μ.

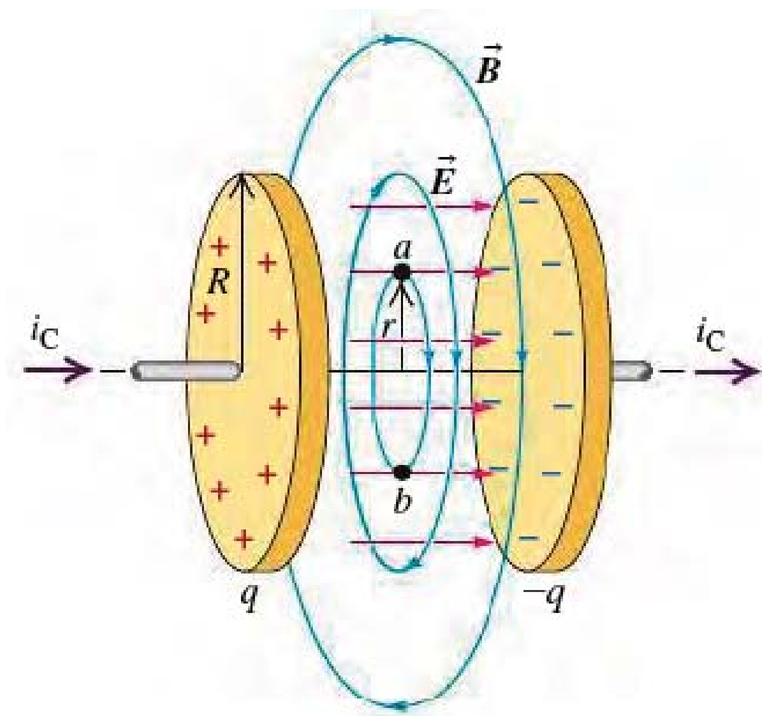
The fictitious displacement current i_D was invented in 1865 by the Scottish physicist James Clerk Maxwell. There is a corresponding *displacement current density* $j_D = i_D/A$; using $\Phi_E = EA$ and dividing Eq.(**??**) by A, we find

$$j_D = \epsilon \frac{dE}{dt} \tag{5}$$

• The Reality of Displacement Current

You might well ask at this point whether displacement current has any real physical significance or whether it is just a ruse to satisfy Ampere's law. Here's a fundamental experiment that help to answer that question.

We take a plane circular area between the capacitor plates. If displacement current really plays the role in Ampere's law that we have claimed, then there ought to be a magnetic field in the region between the plates while the capacitor is charging. We can use our generalized Ampere's law, including displacement current, to predict what this field should be.



• The Reality of Displacement Current

To be specific, let's picture round capacitor plates with radius R. To find the magnetic field at a point in the region between the plates at a distance r from the axis, we apply Ampere's law to a circle of radius r passing through the point, with $r < R$. The circle passes through points a and b. The total current enclosed by the circle is j_D times its area, or $(i_D/\pi R^2)(\pi r^2)$.

The integral $\oint_C \vec{B} \cdot \vec{dl}$ in Ampere's law is just B times the circumference $2\pi r$ of the circle, and because $i_D = i_C$ for the charging capacitor, Ampere's law become

$$\oint_C \vec{B} \cdot \vec{dl} = 2\pi r B = \mu_0 \frac{r^2}{R^2} i_C \tag{6}$$

or

$$B = \frac{\mu_0}{2\pi} \frac{r}{R^2} i_C \tag{7}$$

This result predict that in the region between the plates $\vec{B}$ is zero at the axis and increases linearly with distance from the axis. A similar calculation shows that *outside* the region between the plates (that is, for $r > R$), $\vec{B}$ is the same as though the wire were continuous and the plates no present at all.

When we *measure* the magnetic field in this region, we find that it really is there and that it behaves just as Eq.(**??**) predict. This confirm directly the role of displacement current as a source of magnetic field. It is now established beyond reasonable doubt that Mexwell's displacement current, far from being just an artifice, is a fundamental fact of nature.